2015-10-05 Problems

Introduction to Number Theory

Reference:
"Number Theory 1: Fermat's dream" Kato, Kurokawa, Saito, Chapter 3.

0. We set $h_1(t) = \frac{1+t}{2(1-t)}$ and for $r \geq 2$, set

$$h_r(t) = \left(t\frac{d}{dt}\right)^{r-1} h_1(t).$$

0.1. Compute $h_2(t)$.
0.2. Compute $h_3(t)$.
0.3. Show that $h_r(t) \in \mathbb{Q}[t, \frac{1}{1-t}]$.

1. Use the formula (theorem) discussed during the lecture.
1.1. Show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.$$

1.2. Show that

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots = \frac{\pi}{3\sqrt{3}}.$$

1.3. Show that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots = \frac{\pi^3}{32}.$$

1.4. Show that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \cdots = \frac{\pi^4}{90}.$$

1.5. Compute

$$(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7}) + (\frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15}) + \cdots .$$

1.6. Compute

$$(1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2}) + (\frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{15^2}) + \cdots .$$

2. In this exercise, we prove the formula

$$(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7})+(\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\frac{1}{15})+\cdots=\frac{1}{\sqrt{2}}\log(1+\sqrt{2}).$$

Note that the formula given during the lecture does not apply.
2.1. Let $\zeta_8 = e^{\frac{2\pi i}{8}}$ denote an 8-th root of unity. Show that $\zeta_8 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.
2.1. For $a = 1, 3, 5, 7$, set

$$s_a = \sum_{n=1}^{\infty} \frac{\zeta_8^{an}}{n} = -\log(1-\zeta_8^a).$$

Compute $s_1 - s_3 - s_5 + s_7$ in two ways (one using log and the other using the infinite sum). Compare them to obtain the formula.

3. The Bernoulli numbers B_n $(n = 0, 1, \dots)$ are defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Later in the lecture, we show that for an integer $r > 2$,

$$\zeta(1-r) = -\frac{1}{r} B_r.$$

We can compute B_n explicitly by comparing the coefficients in the equation:

$$\begin{aligned}\frac{x}{e^x-1} &= \frac{x}{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots} \\ &= \frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots} \\ &= 1-\left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)+\left(\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)^2-\cdots\end{aligned}$$

Note that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$.

3.1. Compute B_4 and B_5.
3.2. Compute B_6 and B_7.
3.3. Compute B_8.

3.4. Compute B_{10}.

3.5. Show that

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n - \sum_{n=0}^{\infty} \frac{B_n}{n!} (-x)^n = x.$$

Conclude that $B_n = 0$ if n is odd and > 1.