2015-11-30 Introduction to number theory Problems

References:
Kato-Kurokawa-Saito, “Number Theory 2: Introduction to Class Field Theory”, American Mathematical Society.
Cassels “Global fields”, Chapter 2 in Algebraic Number Theory, ed. Cassels, Frohlich.

1. Let K be a locally compact field.
1.1. Show that K with addition is a locally compact group.
1.2. Show that $K^\times = K \setminus \{0\}$ with multiplication is a locally compact group.

2.1. Show that the restricted product of locally compact groups is a locally compact group. It follows that the adeles $\mathbb{A}_K$ and the ideles $\mathbb{A}_K^\times$ are locally compact topological groups.
2.2. Show that $\mathbb{A}_K$ is a topological ring.
2.3. Show that the set of invertibles of $\mathbb{A}_K$ is the ideles $\mathbb{A}_K^\times$.

3.1. Let p be prime. Show (directly) that $(\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$ is compact.
3.2. Show that $\mathbb{Z}[\sqrt{2}]$ is dense in $\mathbb{R}$.
3.3. Let p be prime. Show directly that $\mathbb{Z}[\frac{1}{p}] \subset \mathbb{Q}_p \times \mathbb{R}$ is discrete.

4.1. Show that a discrete compact topological space is finite.
4.2. Let $f : X \to Y$ be a surjective continuous map between topological spaces. Suppose X is compact. Show that Y is compact.
4.3. Let G be a topological group, H be a subgroup. Equip G/H with the quotient topology.
4.3.1. Show that H is open if G/H is discrete.
4.3.2. Show that H is open only if G/H is discrete.
4.3.3. Show that H is closed if G/H is Hausdorff.
4.3.4. Show that H is closed only if G/H is Hausdorff.

5. Let $n \in \mathbb{N}$. We set $a(n) \in \mathbb{A}_\mathbb{Q}$ to be the element such that $a(n)_\infty = 1 \in \mathbb{R}$ and $a(n)_p = n! + 1 \in \mathbb{Q}_p$ for primes p.
5.1. Show that $a(n)$ tends to 1 in $\mathbb{A}_\mathbb{Q}$ as $n \to \infty$.
5.2. Show that the sequence $a(n)$ does not converge in $\mathbb{A}_\mathbb{Q}^\times$.

6.1. Show that the map $\mathbb{A}_K^\times \to \mathbb{R}^\times$ which sends x to $|x|$ is continuous.
6.2. Show that the map $\mathbb{A}_K \to K_v$ which sends x to $|x|_{K_v}$ is continuous for a finite place v.
6.3. Show that the map $\mathbb{A}_K \to \mathbb{R}$ which sends x to $|x|$ is not continuous.

Proposition 1 (6.78)**.** *1.* $K \subset \mathbb{A}_K$ *is discrete.*
2. $\mathbb{A}_K/K$ *is compact.*

Proposition 2 (6.81)**.** *Let* $a \in K^\times$*. Then* $|a| = 1$*.*

Theorem 1 (6.82)**.** *1.* $K^\times \subset \mathbb{A}_K^\times$ *is discrete.*
2. $C_K^1 = \mathbb{A}_K^1/K^\times$ *is compact.*

Let S be a set of places of K, which contains all the infinite places. Set

$$\mathcal{O}_S := \{x \in K \mid x \in \mathcal{O}_v \text{ for } v \notin S\}.$$

Let $R_S : \mathcal{O}_S^\times \to \prod_{v\in S} \mathbb{R}$ denote the group homomorphism that sends x to $(\log |x|_{K_v})_{v\in S}$. Set

$$\left(\prod_{v\in S} \mathbb{R}\right)^0 := \left\{(e_v)_{v\in S} \in \prod_{v\in S} \mathbb{R} \;\middle|\; \sum_{v\in S} c_v = 0\right\}$$

Proposition 6.81 implies that $R_S(\mathcal{O}_S^\times) \subset (\prod_{v\in S} \mathbb{R})^0$.

Proposition 3 (6.83)**.** *1.* $R_S(\mathcal{O}_S^\times) \subset (\prod_{v\in S} \mathbb{R})^0$ *is discrete.*
2. $(\prod_{v\in S} \mathbb{R})^0/R_S(\mathcal{O}_S^\times)$ *is compact.*
3. $\operatorname{Ker} R_S$ *is finite*

Theorem 2 (6.86)**.** *Set* $r = |S| - 1$ *if* $S \neq \emptyset$ *and* $r = 0$ *if* $S = \emptyset$*. Then*

$$\mathcal{O}_S^\times \cong \mathbb{Z}^r \oplus H$$

for some finite abelian group H*.*

Theorem 3. *The ideal class group* $\mathrm{Cl}(K)$ *is finite.*