2015-12-07 Introduction to number theory

References:
Kato-Kurokawa-Saito, "Number Theory 2: Introduction to Class Field Theory", American Mathematical Society.
Cassels "Global fields", Chapter 2 in Algebraic Number Theory, ed. Cassels, Frohlich.

CORRECTION: Problem 5 in 2015-11-30 should read
5. Let $n \in \mathbb{N}$. We set $a(n) \in \mathbb{A}_{\mathbb{Q}}$ to be the element such that $a(n)_\infty = 1 \in \mathbb{R}$ and $a(n)_p = n! + 1 \in \mathbb{Q}_p$ for primes p.
5.1. Show that $a(n)$ tends to 1 in $\mathbb{A}_{\mathbb{Q}}$ as $n \to \infty$.
5.2. Show that the sequence $a(n)$ does not converge in $\mathbb{A}_{\mathbb{Q}}^\times$.

1. Let G be a Hausdorff topological group. Let H be a subgroup, which is discrete with respect to the subspace topology. Show that H is a closed subgroup of G.

2. Let μ be the invariant measure on $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$. Compute $\mu(p\mathbb{Z}_p)$. Give details.

3. Show that $\mathbb{Z}_p^\times$ is compact.

4. Let K be a global field. Show that the canonical inclusion $\mathbb{A}_K^\times \to \mathbb{A}_K$ is continuous.

Problems for the final exam
Older problem sheets are available at `http://math.hse.ru/number_theory_2015`
1. Problems 2 from 2015-11-02. Possibly for $\mathbb{Z}[\sqrt{-1}]$ instead of $\mathbb{Z}[\sqrt{-2}]$.
2. Problems 4 from 2015-11-09.
3. Problems 3 from 2015-11-16.
4. Problems 8 from 2015-11-16.
5. Problems 3 from 2015-11-23.
6. Problems 3 from 2015-11-30.
7. Problems 6.1-6.3 from 2015-11-30.
8. Problem 2 from 2015-12-07.
9. Problem 3 from 2015-12-07.

Lemma 1 (6.98)**.** *Let G_1, G_2, G_3 be topological abelian groups. Let $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be continuous group homomorphisms. If f and g are isomorphisms modulo compact, then $g \circ f : G_1 \to G_3$ is an isomorphism modulo compact.*

Lemma 2 (6.99)**.** *Let G_1, G_2 be topological abelian groups. Let $f : G_1 \to G_2$ be a continuous (group) homomorphism. Let $H \subset G_1$ be an open subgroup. If f is an isomorphism modulo compact, then the map*

$$f^{-1}(H) \to H$$

that sends $x \in f^{-1}(H)$ to $f(x)$ is an isomorphism modulo compact.

Lemma 3 (6.104)**.** *Let G be a locally compact abelian group and H be its closed subgroup. (Then H and G/H are locally compact.) Let $\alpha : G \xrightarrow{\cong} G$ be an automorphism of topological abelian group. Assume that α induces an isomorphism $H \xrightarrow{\cong} H$. We consider three modules: $|\alpha|_G$, $|\alpha|_H$ and $|\alpha|_{G/H}$. The following equality*

$$|\alpha|_G = |\alpha|_H \cdot |\alpha|_{G/H}$$

holds.

Lemma 4. *Let G be a locally compact abelian group which is not compact. Let μ be an invariant measure on G. For any real number $c \in \mathbb{R}$, there exists a compact subset $C \subset G$ such that $\mu(C) > c$.*

Lemma 5. *Let G be a locally compact abelian group. Let $\Gamma \subset G$ be a discrete subgroup. Let μ be an invariant measure on G. Then there exists a unique invariant measure μ' on G/Γ satisfying the following property: Let $C \subset G$ be a compact subset and C' be the image of C in G/Γ. If $C \to C'$ is injective, then $\mu(C) = \mu'(C')$.*