Exercises 2. Elements of functional analysis and Lagrange multipliers

Calc. of Var. and Op. Co., bachelor 3-4 year, **18.02.2016**

Deadline — March 10, 2016.

2⋄1 Count Fréchet derivatives:
(a) $f : \mathbf{H} \to (x,x)$, where $\mathbf{H}$ is a Hilbert space;
(b) $f : C([0,1]) \to \mathbb{R}$, $f(x(\cdot)) = \left(\int_0^1 x^2(t)dt\right)^3$;
(c) $f : C([0,1]) \to$, $f(x(t)) = \left(\int_0^1 x^2(t)a(t)dt\right)^3$, $a(t) \in C([0,1])$.

2⋄2 Find points such that the functions $f : \mathbb{R}^n \to \mathbb{R}$ do not have Fréchet derivatives
(a) $f(x) = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$; (b) $f(x) = \max_{1\leqslant j\leqslant n} |x_j|$; (c) $f(x) = \sum_{j=1}^n |x_j|$.

2⋄3 Find linear functionals that do not reaches a maximum value on a sphere in normed spaces:
(a) c_0 ($\|x\| = \max_{k\geqslant 1} |x_k|, \lim_{k\to\infty} x_k = 0$); (b) c ($\|x\| = \sup_{k\geqslant 1} |x_i|, \exists \lim_{k\to\infty} x_k$);
(c) l_1 ($\|x\| = \sum_{k\geqslant 1} |x_k|$); (d) $L_1[0,1]]$ ($\|x\| = \int_0^1 |x(t)|dt$);
(e) $C[0,1]$ ($\|x\| = \max_{t\in[0,1]} |x(t)|$).

2⋄4 Solve the finite-dimensional problems by means of the method of Lagrange multipliers:
(a) find the maximal volume of the box, which is made of the sheet of paper of area S;
(b) find a discrete random variable χ with n values which has the maximal entropy

$$H(\xi) = \sum_{i=1}^n p_i \ln \frac{1}{p_i}.$$

(c) Prove the Holder's inequality

$$\left(\frac{1}{n}\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{n}\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}, \quad 0 < p \leqslant q \leqslant \infty.$$

(d) $xyz \to extr$, $x^2 + y^2 + z^2 \leqslant 1$.