

# Elliptic Functions

Applications of elliptic integrals

## §2.1 Arithmetic-geometric mean

Well-known fact: geometric mean  $\leq$  arithmetic mean:

$$\sqrt{ab} \leq \frac{a+b}{2} \quad (\text{G} < \text{A})$$

for  $a, b > 0$ .

Repeat the process:  $a_0 = a, b_0 := b$  for  $a \geq b > 0$ ,

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}.$$

Because of the inequality (G < A),

$$b_n \leq b_{n+1} \leq a_{n+1} \leq a_n,$$

i.e.,

$$b = b_0 \leq b_1 \leq \cdots \leq b_n \leq \cdots \leq a_n \leq \cdots \leq a_1 \leq a_0 = a.$$

$\{a_n\}_{n=0,1,2,\dots}, \{b_n\}_{n=0,1,2,\dots}$ : bounded monotone sequences.

$$\implies \exists \alpha := \lim_{n \rightarrow \infty} a_n, \beta := \lim_{n \rightarrow \infty} b_n.$$

In fact,  $\alpha = \beta$ , i.e.,  $c_n := \frac{1}{2}(a_n - b_n) \rightarrow 0$ .

Proof:

$$0 \leq a_n - b_n \leq a_n - b_{n-1} = \frac{a_{n-1} + b_{n-1}}{2} - b_{n-1} = c_{n-1}.$$

$$\implies 0 \leq c_n \leq \frac{c_{n-1}}{2} \leq \dots \leq \frac{c_0}{2^n} \rightarrow 0 \quad (n \rightarrow \infty). \quad \square$$

$M(a, b) := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$  : *arithmetic-geometric mean (AGM)*.

Easy to see:

(i)  $M(a, b) = M(b, a)$ .

(ii)  $M(\lambda a, \lambda b) = \lambda M(a, b)$ .

(iii)  $M(a, a) = a$ .

(iv)  $M(a, b) = M(a_n, b_n)$  ( $n = 0, 1, 2, \dots$ ).

(ii)  $\implies$  enough to study  $M(a, 1)$  or  $M(1, b)$  for  $a > 1 > b$ .

Theorem: If  $0 < k < 1$ , then  $M(1, k) = \frac{\pi}{2} \frac{1}{K(k')}$ , where

- $k' := \sqrt{1 - k^2}$ ,

- $K(k')$ : complete elliptic integral of the first kind,

$$K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}} = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k'^2 z^2)}}.$$

Proof:

Lemma: for  $a \geq b > 0$ ,

$$I(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(a, b)}.$$

Once this lemma is proved, the theorem is its corollary:

$$M(1, k) = \frac{\pi}{2} \frac{1}{I(1, k)},$$

and

$$\begin{aligned} I(1, k) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta + k^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - k^2) \sin^2 \theta}} = K(k'). \end{aligned}$$

□

Proof of the lemma:

Key:

$$I(a, b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

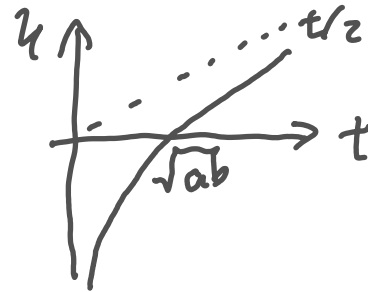
In fact,

$$\begin{aligned} I(a, b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\cos \theta \sqrt{a^2 + b^2 \tan^2 \theta}} \\ &= \int_0^{\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}}, \end{aligned} \left( \begin{array}{l} t = b \tan \theta, \quad \frac{dt}{d\theta} = \frac{b}{\cos^2 \theta}, \\ b^2 + t^2 = \frac{b^2}{\cos^2 \theta} \end{array} \right).$$

Another change of the integration variable:  $u = \frac{1}{2} \left( t - \frac{ab}{t} \right)$ ,

$$\frac{du}{dt} = \frac{1}{2} \left( 1 + \frac{ab}{t^2} \right) = \frac{\sqrt{ab + u^2}}{t}, \quad (a^2 + t^2)(b^2 + t^2) = t^2((a + b)^2 + 4u^2).$$

(Figure: graph of  $u = u(t)$ .)



$$\begin{aligned} I(a, b) &= \int_{-\infty}^{\infty} \frac{du}{\sqrt{((a + b)^2 + 4u^2)(ab + u^2)}} \\ &= 2 \int_0^{\infty} \frac{du}{\sqrt{((a + b)^2 + 4u^2)(ab + u^2)}} \\ &= \int_0^{\infty} \frac{du}{\sqrt{\left(\left(\frac{a+b}{2}\right)^2 + u^2\right) \left((\sqrt{ab})^2 + u^2\right)}} \end{aligned}$$

Comparing the expression of  $I(a, b)$  in the last two pages, we have

$$I(a, b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

Repeating the procedure,

$$I(a, b) = I(a_1, b_1) = \cdots = I(a_n, b_n) = \cdots \longrightarrow I(M(a, b), M(a, b)).$$

In general,

$$\begin{aligned} I(c, c) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{c^2 \cos^2 \theta + c^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{c\sqrt{\cos^2 \theta + \sin^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{c} = \frac{\pi}{2c}. \end{aligned}$$

$$\implies I(a, b) = \frac{\pi}{2M(a, b)}.$$

□



Similarly, when  $k > 1$ ,

$$M(k, 1) = \frac{\pi}{2} \frac{1}{K(k')}, \quad k' := \sqrt{1 - k^2}.$$

Remark:  $k' \notin \mathbb{R}$  but  $K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}$  makes sense.

Example:  $k = \sqrt{2}$  ( $k' = \sqrt{-1}$ ):  $K(\sqrt{-1}) = \frac{\pi}{2M(\sqrt{2}, 1)}$ , i.e.,

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{\pi}{2M(\sqrt{2}, 1)}.$$

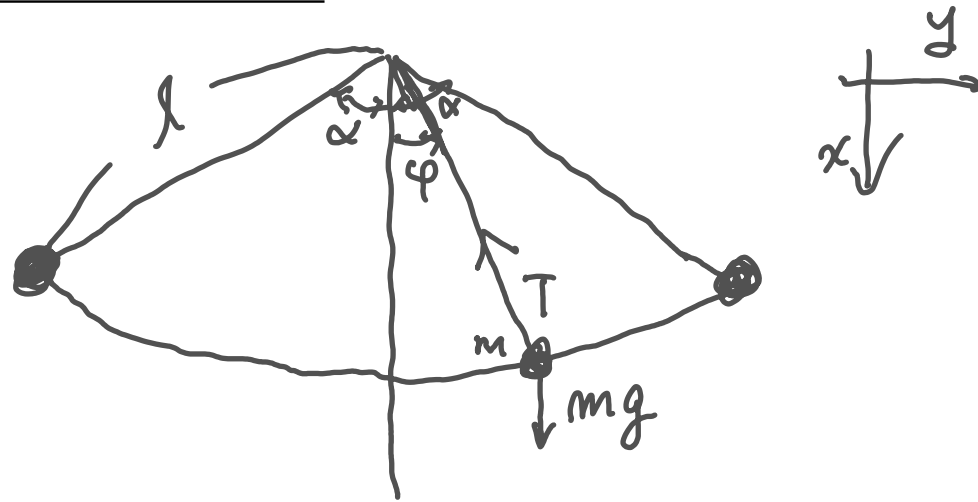
Gauß found this formula by computing the both sides *by hand!*

From  $a_4$  and  $b_4$ :  $M(\sqrt{2}, 1) = 1.981402347\dots = \frac{\pi}{2 \int_0^1 \frac{dx}{\sqrt{1 - x^4}}}$ .

(cf. Gauß, Gesammelte Werke, Tagebuch.)

## §2.2 Motion of a simple pendulum

(Figure of a simple pendulum.)



$l$ : length of the string,  $T$ : tension,  $mg$ : gravity.

$(x(t), y(t)) = (l \cos \varphi(t), l \sin \varphi(t))$ : coordinates of the point mass.

$$\begin{aligned} \text{acceleration} &= \frac{d^2}{dt^2} (l \cos \varphi(t), l \sin \varphi(t)) \\ &= \frac{d}{dt} (-l\dot{\varphi} \sin \varphi, l\dot{\varphi} \cos \varphi) \\ &= (-l\ddot{\varphi} \sin \varphi - l\dot{\varphi}^2 \cos \varphi, l\ddot{\varphi} \cos \varphi - l\dot{\varphi}^2 \sin \varphi) \end{aligned}$$

The equation of motion:

$$ml \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} \ddot{\varphi} - ml \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} \dot{\varphi}^2 = \begin{pmatrix} -T \cos \varphi + mg \\ -T \sin \varphi \end{pmatrix}.$$

$x$ -component  $\times (-\sin \varphi) + y$ -component  $\times \cos \varphi$ :  $ml\ddot{\varphi} = -mg \sin \varphi$ , i.e.,

$$\frac{d^2 \varphi}{dt^2} = -\omega^2 \sin \varphi, \quad \omega := \sqrt{\frac{g}{l}}.$$

Linear approximation: amplitude  $\ll 1 \implies \sin \varphi \approx \varphi$ .

$$\frac{d^2 \varphi}{dt^2} = -\omega^2 \varphi.$$

A *LINEAR* differential equation of the second order.

General solution:  $\varphi(t) = c_1 \cos \omega t + c_2 \sin \omega t$ ,  $c_1, c_2 \in \mathbb{R}$ .

Exact solutions without approximation  $\sin \varphi \approx \varphi$ .

(eq. of motion)  $\times \dot{\varphi}$ :

$$\frac{d^2 \varphi}{dt^2} \frac{d\varphi}{dt} = -\omega^2 \sin \varphi \frac{d\varphi}{dt},$$

$$\frac{1}{2} \frac{d}{dt} \left( \frac{d\varphi}{dt} \right)^2 = \frac{d}{dt} (\omega^2 \cos \varphi)$$

$$\xrightarrow{\int(\dots)dt} \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 = \omega^2 \cos \varphi + (\text{const.}),$$

$$\tilde{E} := \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 - \omega^2 \cos \varphi = (\text{const.}),$$

$\approx$  conservation law of the energy.

Remark: the total energy of the pendulum is

$$(\text{kinetic energy}) + (\text{potential energy}) = \frac{ml^2}{2} \left( \frac{d\varphi}{dt} \right)^2 - mgl \cos \varphi = ml^2 \tilde{E}.$$

$\alpha :=$  maximum amplitude, i.e., if  $\varphi(t_0) = \alpha$ ,  $\dot{\varphi}(t_0) = 0$ .

$\implies \tilde{E} = -\omega^2 \cos \alpha$ , hence

$$\frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 = \omega^2 (\cos \varphi - \cos \alpha) = 2\omega^2 \left( \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\varphi}{2} \right),$$

$$\frac{d\varphi}{dt} = 2\omega \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\varphi}{2}}.$$

Note:  $\left| \sin \frac{\varphi}{2} \right| < \sin \frac{\alpha}{2}$ , because  $|\varphi| \leq \alpha < \pi$ .

Notations:  $k := \sin \frac{\alpha}{2}$ ,  $\theta := \arcsin \left( k^{-1} \sin \frac{\varphi}{2} \right)$ , i.e.,  $\sin \frac{\varphi}{2} = k \sin \theta$ .

The equation of motion becomes

$$\frac{d\varphi}{dt} = 2\omega \sqrt{k^2 - k^2 \sin^2 \theta} = 2k\omega \cos \theta.$$

On the other hand,  $\frac{d\varphi}{dt} = \frac{d\varphi}{d\theta} \frac{d\theta}{dt}$  and

$$\frac{d}{d\theta} k \sin \theta = \frac{d}{d\theta} \left( \sin \frac{\varphi}{2} \right), \text{ i.e., } k \cos \theta = \frac{\cos \frac{\varphi}{2}}{2} \frac{d\varphi}{d\theta} = \frac{\sqrt{1 - k^2 \sin^2 \theta}}{2} \frac{d\varphi}{d\theta}$$

Hence,

$$\frac{2k \cos \theta}{\sqrt{1 - k^2 \sin^2 \theta}} \frac{d\theta}{dt} = 2k\omega \cos \theta,$$

$$\frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \frac{d\theta}{dt} = \omega,$$

$$\int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^t \omega dt = \omega t.$$

(Normalisation:  $\theta(0) = 0$ , i.e.,  $\varphi(0) = 0$ .)

The left hand side =  $F(k, \theta(t))$  = elliptic integral of the first kind.

The motion of the pendulum:

$$t(\theta) = \frac{1}{\omega} F(k, \theta) = \sqrt{\frac{l}{g}} F\left(\sin \frac{\alpha}{2}, \theta\right).$$

The period = 4(time between  $\varphi = 0$  and  $\varphi = \alpha$ )

$$= 4(\text{time between } \theta = 0 \text{ and } \theta = \frac{\pi}{2}) = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\alpha}{2}\right).$$

(The period depends on the amplitude!)