Elliptic Functions

Classification of elliptic integrals

§1.3 Classification of elliptic integrals

Recall: Indefinite integrals of the form

$$\int R(x)\,dx, \qquad R(x) : \text{rational function}$$

can be expressed in terms of elementary functions:

$$\text{rational functions},\ \log,\ \arctan\,.$$

Indefinite integrals of the form

$$\int R(x, \sqrt{\varphi(x)})\,dx,$$

$$R(x,s) : \text{rational function},\ \ \varphi(x) : \text{quadratic polynomial}$$

can be reduced to an integral of rational functions, using trigonometric functions and their inverse.

Examples:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{d\sin\theta}{\cos\theta} \qquad (x = \sin\theta)$$

$$= \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta = \arcsin x.$$

$$\int \sqrt{1+x^2}\, dx = 2\int \frac{(1+t^2)^2}{(1-t^2)^2}\, dt \qquad \left(x = \frac{2t}{1-t^2}\right)$$

$$= \cdots$$

$$= \frac{1}{2}\Big(x\sqrt{1+x^2} + \log(x + \sqrt{1+x^2})\Big).$$

If $\deg \varphi(x) \geqq 3$, in general, integrals $\displaystyle\int R(x, \sqrt{\varphi(x)})\, dx$ cannot be expressed in terms of elementary functions.

Note: for simplicity, coefficients of $R(x, s)$ and $\varphi(x) \in \mathbb{C}$. For example,

$$\arctan x = \int \frac{dx}{1+x^2} = \frac{1}{2i} \int \left(\frac{1}{x-i} - \frac{1}{x+i} \right) dx = \frac{\log(x-i) - \log(x+i)}{2i}.$$

$\Longrightarrow$ In $\mathbb{C}$, we do not need $\arctan$.

Definition:

$\varphi(x)$: polynomial of degree 3 or 4 without multiple roots,

$R(x, s)$: rational function,

$\Longrightarrow$ integrals of the form $\displaystyle\int R(x, \sqrt{\varphi(x)})\, dx$ are called the *elliptic integrals*.

(When $\deg \varphi(x) \geqq 5$: *hyperelliptic* integrals.)

The case $\deg \varphi(x) = 3$ and $\deg \varphi(x) = 4$ are essentially the same!

Example: $\deg \varphi(x) = 3$.

$$\varphi(x) = a(x-\alpha_1)(x-\alpha_2)(x-\alpha_3),$$

α_1, α_2, α_3: distinct by assumption.

Take a fractional linear transformation: $x = T(y) = \dfrac{Ay+B}{Cy+D}$, such that

- $C + D = 0$, i.e., $T(1) = \infty$.
- $T(\infty) = A/C \neq \alpha_i$ $(i = 1, 2, 3)$, i.e., $\beta_i := T^{-1}(\alpha_i) \neq \infty$.

$$\Longrightarrow \quad x - \alpha_i = \frac{Ay+B}{Cy+D} - \frac{A\beta_i + B}{C\beta_i + D}$$
$$= \text{const.} \times \frac{y-\beta_i}{y-1}. \qquad \left(\text{const.} = \frac{A - \alpha_i C}{C}\right)$$

$$\begin{aligned}
R(x,\sqrt{\varphi(x)}) &= R\left(\frac{Ay+B}{Cy+D}, \text{const.}\sqrt{\frac{(y-\beta_1)(y-\beta_2)(y-\beta_3)}{(y-1)^3}}\right) \\
&= R\left(\frac{Ay+B}{Cy+D}, \text{const.}\frac{\sqrt{(y-\beta_1)(y-\beta_2)(y-\beta_3)(y-1)}}{(y-1)^2}\right) \\
&= \tilde{R}(y,\sqrt{(y-\beta_1)(y-\beta_2)(y-\beta_3)(y-1)}).
\end{aligned}$$

($\tilde{R}(y,t)$: new rational function) and

$$dx = \frac{dx}{dy}dy = \frac{AD-BC}{(Cy+D)^2}dy.$$

$$\Longrightarrow \quad \int R(x,\sqrt{\varphi(x)})\,dx = \int \tilde{\tilde{R}}(y,\sqrt{\psi(y)})\,dy,$$

$$\psi(y) = (y-\beta_1)(y-\beta_2)(y-\beta_3)(y-1),\ \tilde{\tilde{R}}(y,t) = \tilde{R}(y,t) \times \frac{AD-BC}{(Cy+D)^2}.$$

Exercise:

$$\int R(x, \sqrt{\varphi(x)})\, dx \ (\deg \varphi = 4) \rightsquigarrow \int R'(y, \sqrt{\psi(y)})\, dy \ (\deg \psi = 3).$$

Hereafter $\underline{\deg \varphi(x) = 4}$.

Note:

- $\varphi(x) = \prod_{i=1}^{4} (x - \alpha_i)$ has four parameters $(\alpha_1, \ldots, \alpha_4)$.
- fractional linear transformations $\dfrac{Ax + B}{Cx + D}$ $(AD - BC = 1)$ determined by three parameters.

$\Longrightarrow$ remains four $-$ three $=$ one parameter.

In fact, may assume

$$\varphi(x) = \varphi_k(x) = (1 - x^2)(1 - k^2 x^2),$$

by using a fractional linear transformation T, such that

$$(T\alpha_1, T\alpha_2, T\alpha_3, T\alpha_4) = (1, k^{-1}, -1, -k^{-1}).$$

<u>Exercise</u>:

(i) Express k in terms of the cross ratio $\lambda = \dfrac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}$ of $(\alpha_1, \ldots, \alpha_4)$.

(ii) Show that such T exists.

(iii) $\displaystyle\int R(x, \sqrt{\varphi(x)})\, dx \rightsquigarrow \int R_k(y, \sqrt{(1 - y^2)(1 - k^2 y^2)})\, dy.$

Theorem (Legendre-Jacobi standard forms):

Any elliptic integral is a linear combination of

- elementary functions (combinations of rational functions, $\log$, inverse trigonometric functions and $\sqrt{\varphi(x)}$),
- $\displaystyle\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ (*elliptic integral of the first kind*),
- $\displaystyle\int \sqrt{\frac{1-k^2x^2}{1-x^2}}dx = \int \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$ (*elliptic integral of the second kind*),
- $\displaystyle\int \frac{dx}{(x^2-\alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}dx$, α: parameter (*elliptic integral of the third kind*). □

k: modulus of the elliptic integral.

Proof:

May assume $\varphi(x) = \sqrt{(1-x^2)(1-k^2x^2)}$.

$s := \sqrt{\varphi(x)}$, i.e., $s^2 = \varphi(x)$.

$R(x,s)$: rational function $\Longrightarrow \exists$ polynomials $P_1(x), P_2(x), Q_1(x), Q_2(x)$,

$$\begin{aligned}
R(x, \sqrt{\varphi(x)}) &= \frac{P_1(x) + P_2(x)s}{Q_1(x) + Q_2(x)s} \\
&= \frac{(P_1(x) + P_2(x)s)(Q_1(x) - Q_2(x)s)}{(Q_1(x) + Q_2(x)s)(Q_1(x) - Q_2(x)s)} \\
&= \frac{\tilde{P}_1(x) + \tilde{P}_2(x)s}{Q_1(x)^2 - Q_2(x)^2\varphi(x)} \qquad (\tilde{P}_1(x), \tilde{P}_2(x) : \text{ polynomials}) \\
&= R_1(x) + \tilde{R}_2(x)s \qquad (R_1(x), R_2(x) : \text{ rational functions}) \\
&= R_1(x) + \frac{R_2(x)}{s} \qquad (R_2(x) = \tilde{R}_2(x)\varphi(x)).
\end{aligned}$$

$$\int R(x, \sqrt{\varphi(x)})\, dx = \int R_1(x)\, dx + \int \frac{R_2(x)}{s}\, dx.$$

Since the rational function $R_2(x)$ is expanded as

$$R_2(x) = \text{(polynomial)} + \sum_{j=1}^{M} \sum_{n=1}^{n_j} \frac{a_{jn}}{(x-\alpha_j)^n},$$

the integral $\int R(x, \sqrt{\varphi(x)})\, dx$ is a linear combination of

- the integral of a rational function $R_1(x)$,
- $I_n := \int \frac{x^n}{s}\, dx$ $(n = 0, 1, 2, \dots)$,
- $J_n(\alpha) := \int \frac{dx}{(x-\alpha)^n s}$ $(n = 0, 1, 2, \dots)$.

Recurrence relations:

- recurrence relations among I_n's: integrate the relation,

$$
\begin{aligned}
\frac{d}{dx}x^n s &= nx^{n-1}s + \frac{x^n}{2s}\frac{d\varphi}{dx} \\
&= \frac{nx^{n-1}\varphi(x)}{s} + \frac{x^n(\text{polynomial of } \deg = 3)}{s}
\end{aligned}
$$

$$
\Longrightarrow x^n s = \begin{cases} c_{n+3}I_{n+3} + \cdots + c_n I_n + c_{n-1}I_{n-1}, & (n \neq 0), \\ c_3 I_3 + \cdots + c_0 I_0, & (n = 0). \end{cases}
$$

$\xrightarrow{\text{induction}}$ I_n $(n \geqq 3)$ = a linear combintion of (polynomial)$\times s$, I_2, I_1, I_0.

• recurrence relations among $J_n(\alpha)$'s: For $n \geqq 1$,

$$\frac{d}{dx}\frac{s}{(x-\alpha)^n} = \frac{-ns}{(x-\alpha)^{n+1}} + \frac{1}{2(x-\alpha)^n s}\frac{d\varphi}{dx}$$
$$= \frac{1}{(x-\alpha)^{n+1}s}\left(-n\varphi(x) + \frac{x-\alpha}{2}\frac{d\varphi}{dx}\right)$$
$$= \frac{1}{(x-\alpha)^{n+1}s}\sum_{i=0}^{4} d_{n,i}(x-\alpha)^i,$$

where $d_{n,0} = -n\varphi(\alpha)$, $d_{n,1} = \left(\frac{1}{2} - n\right)\varphi'(\alpha)$.

Integrating this relation we have

$$\frac{s}{(x-\alpha)^n} = d_{n,0}J_{n+1} + \cdots + d_{n,4}J_{n-3}.$$

1. $\varphi(\alpha) \neq 0 \Rightarrow d_{n,0} \neq 0$.

   J_{n+1} is a linear combination of $J_n, \ldots, J_{n-3}$ and $\dfrac{s}{(x-\alpha)^n}$.

   $\xrightarrow{\text{induction}} J_n \ (n \geqq 2) =$ a linear combination of J_1, J_0, J_{-1}, J_{-2} and (rational function)$\times s$.

2. $\varphi(\alpha) = 0 \Rightarrow \varphi'(\alpha) \neq 0$. Hence $d_{n,0} = 0$, $d_{n,1} \neq 0$.

   J_n is a linear combination of $J_{n-1}, \ldots, J_{n-3}$ and $\dfrac{s}{(x-\alpha)^n}$.

   $\xrightarrow{\text{induction}} J_n \ (n \geqq 1) =$ a linear combination of J_0, J_{-1}, J_{-2} and (rational function)$\times s$.

On the other hand, $J_{-2} = \displaystyle\int \frac{(x-\alpha)^2}{s}\,dx$, $J_{-1} = \displaystyle\int \frac{x-\alpha}{s}\,dx$ and $J_0 = \displaystyle\int \frac{dx}{s}$ are linear combinations of I_0, I_1 and I_2.

Summarising, in any case, $\int R(x, \sqrt{\varphi(x)})\, dx$ is a linear combination of integrals of rational functions, (rational function)$\times\sqrt{\varphi(x)}$ and

$$I_0 = \int \frac{dx}{\sqrt{\varphi(x)}},$$

$$I_1 = \int \frac{x\, dx}{\sqrt{\varphi(x)}},$$

$$I_2 = \int \frac{x^2\, dx}{\sqrt{\varphi(x)}},$$

$$J_1(\alpha) = \int \frac{dx}{(x-\alpha)\sqrt{\varphi(x)}}.$$

- I_0: the elliptic integral of the first kind.
- $I_1 = \dfrac{1}{2}\displaystyle\int \frac{dt}{\sqrt{(1-t)(1-kt)}}$ $(x = t^2)$

  $=$ elementary function (inverse trigonometric function).
- $I_2 = \displaystyle\int \frac{\frac{1}{k^2}(1-(1-k^2x^2))}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \frac{1}{k^2}I_0 - \frac{1}{k^2}\int \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx.$

  $=$ (elliptic integral of the first kind) $+$ (the second kind).
- $J_1 = \displaystyle\int \frac{(x+\alpha)\,dx}{(x^2-\alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}$

  $$= \frac{1}{2}\int \frac{dt}{(t-\alpha^2)\sqrt{(1-t)(1-k^2t)}} + \alpha\int \frac{dx}{(x^2-\alpha^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

  $=$ (elementary function) $+$ (elliptic integral of the third kind).

□

Another standard forms (Riemann standard form):

$$\int \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}},$$

$$\int \frac{x\,dx}{\sqrt{x(1-x)(1-\lambda x)}},$$

$$\int \frac{dx}{(x-\alpha)\sqrt{x(1-x)(1-\lambda x)}}.$$