Elliptic Functions

Introduction

§0.1 What is an elliptic function

Elliptic function = Doubly periodic meromorphic function on $\mathbb{C}$.

Too simple object?

Indeed, in most of modern textbooks on the complex analysis, elliptic functions appear usually just as examples.

BUT! in fact, many branches of modern mathematics, especially algebraic geometry, came out of study of elliptic functions in the XIXth century.

§0.2 Subjects related to elliptic functions

Subjects which will and will not appear in the course

- Elliptic integrals:

$$z(u) = \int_{x_0}^{u} R(x, \sqrt{\varphi(x)})\, dx.$$

$R(x, s)$: rational function in two variables.

$\varphi(x)$: polynomial of degree three or four.

Example: arc length of an ellipse.

- Elliptic function

The function $u(z)$ inverse to the elliptic integral $z(u)$

— *an elliptic function*! (discovery of Abel, Jacobi, Gauß)

- Riemann surface

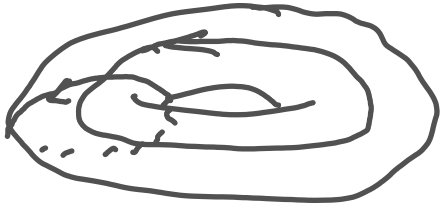
Because of the square root, an elliptic function is "multi-valued".

$\Longrightarrow$ requires "uniformisation"

$\Longrightarrow$ Riemann surface.

- Elliptic curve

Compactification of the Riemann surface of an elliptic integral



$$= \text{elliptic curve } (= \text{torus})$$

$$\text{Periods of the elliptic function} = \int_{\text{loop on the elliptic curve}} R(x, \sqrt{\varphi(x)})\, dx.$$

Elliptic curve $\cong \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. ($\omega_1, \omega_2$: periods.)

- Similarity of elliptic and rational functions.

Elliptic function $=$ meromorphic function on an elliptic curve.

$\Longleftrightarrow$

Rational function $=$ meromorphic function on the Riemann sphere.

$$\text{Rational function} = \frac{\text{polynomial}}{\text{polynomial}} \Longleftrightarrow \text{Elliptic function} = \frac{\theta\text{-function}}{\theta\text{-function}}.$$

- Similarity of elliptic and trigonometric functions.

addition formulae:

Trigonometric: $\sin(x+y) = \sin x \cos y + \sin y \cos x$.

Elliptic: $\operatorname{sn}(x+y) = \dfrac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$.

differential equation:

Trigonometric: $(\sin x)' = \cos x$, $((\sin x)')^2 = 1 - \sin^2 x$.

Elliptic: $(\operatorname{sn} x)' = \operatorname{cn} x \operatorname{dn} x$, $((\operatorname{sn} x)')^2 = (1 - \operatorname{sn}^2 x)(1 - k^2 \operatorname{sn}^2 x)$.

$((\wp(z))')^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$.

- Algebraic geometry and number theory

The differential equation of $\mathrm{sn}(z)$ implies that the image of

$$\mathbb{C} \ni z \mapsto (x = \mathrm{sn}(z), y = \mathrm{sn}'(z)) \in \mathbb{C}^2$$

is an *algebraic curve* $y^2 = \varphi_{\mathrm{sn}}(x) := (1 - x^2)(1 - k^2x^2)$.

The differential equation of $\wp(z)$ implies that the image of

$$\mathbb{C} \ni z \mapsto (x = \wp(z), y = \wp'(z)) \in \mathbb{C}^2$$

is an *algebraic curve* $y^2 = \varphi_{\wp}(x) := 4x^3 - g_2x - g_3$.

Elliptic curve in algebraic geometry: the completion of $y^2 = \varphi(x)$

($\varphi(x) =$ polynomial in x of degee three or four).

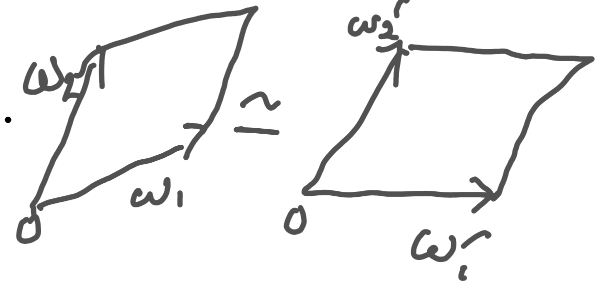
In general (especially in number theory): $\varphi(x) \in k[x]$, $(x, y) \in k^2$.

k: a field, not necessarily $= \mathbb{C}$.

- When are two elliptic curves the “same”?

Elliptic curves $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\mathbb{C}/\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are the same (isomorphic)

$\Longleftrightarrow \exists a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ \omega_2'/\omega_1' = \dfrac{a(\omega_2/\omega_1) + b}{c(\omega_2/\omega_1) + d}$.



$\Longrightarrow$ The theory of the *moduli space* of elliptic curves.

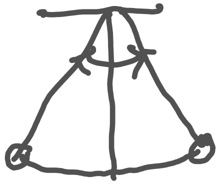
Elliptic curves $y^2 = 4x^3 - g_2x - g_3$ and $y^2 = 4x^3 - g_2'x - g_3'$ (over $\mathbb{C}$)

are the same (isomorphic)

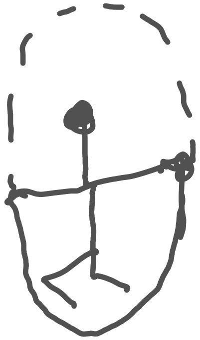
$$\Longleftrightarrow j(g_2, g_3) = j(g_2', g_3'),\ \text{where } j(g_2, g_3) = \frac{(12g_2)^3}{g_2^3 - 27g_3^2}.$$

$\Longrightarrow$ The theory of the *modular functions*.

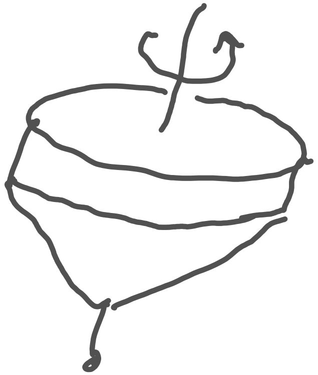
- Application to physics.
  - Mechanics:
    - motion of pendulum.



    - shape of skipping ropes.



    - top (= rotating rigid body with one fixed point).



  - Integrable systems:
    - solutions of Korteweg-de Vries equation, Toda lattice.
    - definition of eight vertex model.

- Various other applications.
  - arithmetic-geometric mean: $a \geqq b > 0$, $a_0 = a$, $b_0 = b$,

    $$a_{n+1} := \frac{a_n + b_n}{2}, b_{n+1} := \sqrt{a_n b_n} \Longrightarrow \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n =: M(a, b)$$

    $M(a, b)$ is expressed by means of an elliptic integral (Gauß).

    $\Longrightarrow$ A fast converging recurrent formula for π can be derived.

  - A formula for roots of quintic equations:

    Abel and Galois proved that "quintic equations are not solvable", i.e.,

    $\nexists$ formula for roots, IF one can use only $\pm, \times, /$ and $\sqrt[n]{}$.

    $\longleftrightarrow$ IF one can use elliptic integrals (and θ-functions),

    $\exists$ a formula for roots!

§0.3 Plan of the course

1. Elliptic integrals (over $\mathbb{R}$).
   - (a) arc length of ellipses, lemniscates, etc.
   - (b) classification of elliptic integrals.
   - (c) applications to mathematics (arithmetic-geometric mean).
   - (d) applications to physics (pendulum and skipping ropes).
2. Elliptic functions (over $\mathbb{R}$)
   - (a) inverse function of elliptic integrals.
   - (b) Jacobi's elliptic functions.
   - (c) properties (addition formulae, differential equations).

3. Complex elliptic integrals.
    (a) differentials (1-forms) and elliptic integrals.
    (b) Riemann surface of irrational (algebraic) functions.
    (c) Compactifications of Riemann surfaces of $y^2 = \varphi(x)$, $\deg \varphi = 3, 4$, i.e., elliptic curves.
    (d) periods and complete elliptic integrals.
    (e) Abel-Jacobi theorem.
4. Elliptic functions (over $\mathbb{C}$)
    (a) doubly periodic meromorphic functions.
    (b) Weierstraß' $\wp$-function and its properties.
    (c) theta functions.
    (d) complex Jacobi functions and their properties.