

Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.

- The rule of evaluation is:

$$(\text{your final mark}) = \min \left\{ \text{integer part of } \frac{3}{2}(\text{total points you get}), 10 \right\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **4 – 6**: 15 February 2016.

4. (1 pt.) Let $R(x, s)$ be a rational function of x and s and $\varphi(x)$ be a polynomial of degree four without multiple roots. Show that the elliptic integral $\int R(x, \sqrt{\varphi(x)}) dx$ is rewritten in the form $\int \tilde{R}(y, \sqrt{\psi(y)}) dy$, where $\tilde{R}(y, s')$ is a rational function in (y, s') and $\psi(y)$ is a polynomial of degree three. (Hint: Use the fractional linear transformation of the variable $x \mapsto y$.)

5. (1 pt.) In the lecture, we claimed that an elliptic integral $\int R(x, \sqrt{\varphi(x)}) dx$ with $\varphi(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ can be transformed into $\int \tilde{R}(y, \sqrt{\varphi_k(y)}) dy$ with $\varphi_k(y) = (1 - y^2)(1 - k^2 y^2)$ ($k \neq 0, \pm 1$) by a fractional linear transformation.

(i) Assuming that such a fractional linear transformation exists, express k in terms of the cross-ratio (the anharmonic ratio) $\lambda = \frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}$ of $\alpha_1, \dots, \alpha_4$.

(ii) Show that such a linear transformation really exists.

6. (1 pt.) Reduce the elliptic integral $\int \frac{dx}{\sqrt{1+x^4}}$ to the standard form.