

Elliptic Functions

Elliptic integrals over \mathbb{R}

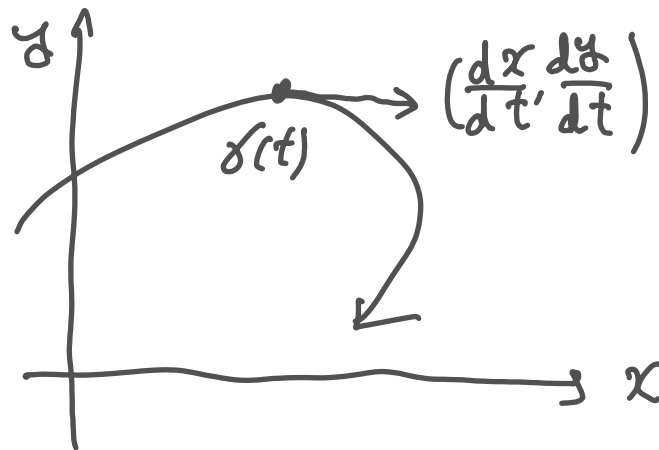
§1.1 Arc length of an ellipse

Everybody has learned that the length of a circle of radius a is $2\pi a$.

How can one *prove* this?

From the analysis course we know the formula for the arc length of a curve:

$\gamma : [a, b] \ni t \mapsto \gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$: a smooth curve.



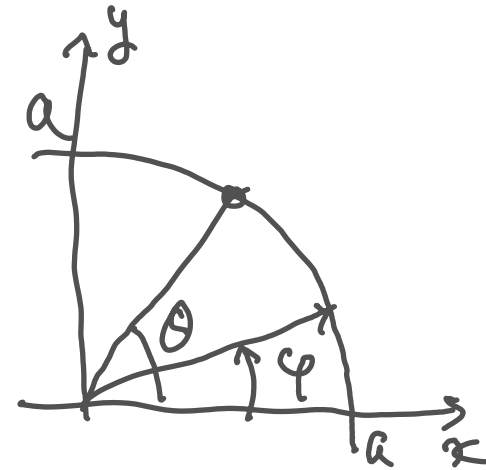
$(x(t), y(t))$: C^1 -class, i.e., $x'(t), y'(t)$ exist and are continuous.)

$$\implies \text{the length of } \gamma = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

(The integrand = speed of the moving point $(x(t), y(t))$.)

Parametrisation of an arc of a circle:

$$(x(\varphi), y(\varphi)) = (a \cos \varphi, a \sin \varphi), \quad (\varphi \in [0, \theta]).$$



$$\begin{aligned} \text{The length of this arc} &= \int_0^\theta \sqrt{\left(\frac{d}{d\varphi} a \cos \varphi\right)^2 + \left(\frac{d}{d\varphi} a \sin \varphi\right)^2} d\varphi \\ &= \int_0^\theta \sqrt{a^2 \sin^2 \varphi + a^2 \cos^2 \varphi} d\varphi = a\theta. \end{aligned}$$

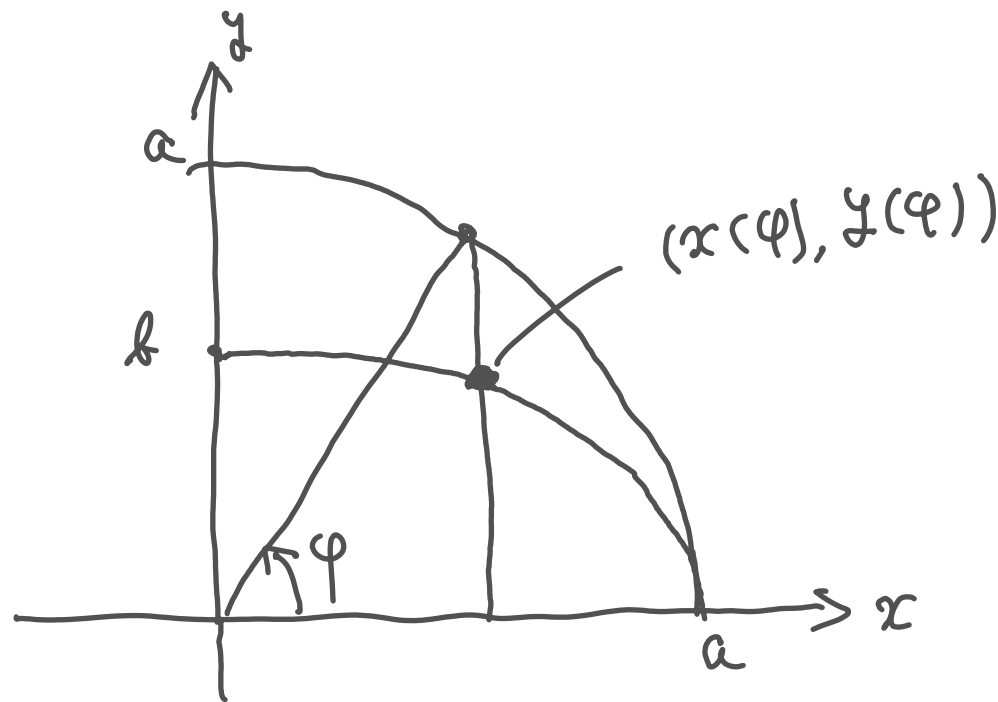
In particular the arc length of the circle = $a \times 2\pi$.

□

How about the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

Parametrisation of an arc of the ellipse:

$$(x(\varphi), y(\varphi)) = (a \sin \varphi, b \cos \varphi) \quad (\varphi \in [0, \theta]).$$



$$\begin{aligned}
\text{The length of this arc} &= \int_0^\theta \sqrt{\left(\frac{d}{d\varphi} a \sin \varphi\right)^2 + \left(\frac{d}{d\varphi} b \cos \varphi\right)^2} d\varphi \\
&= \int_0^\theta \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} d\varphi \\
&= a \int_0^\theta \sqrt{1 - \frac{a^2 - b^2}{a^2} \sin^2 \varphi} d\varphi \\
&= a \int_0^\theta \sqrt{1 - k^2 \sin^2 \varphi} d\varphi,
\end{aligned}$$

where $k := \sqrt{(a^2 - b^2)/a^2}$ } *modulus.*
eccentricity

$$E(k, \theta) := \int_0^\theta \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi$$

— incomplete *elliptic integral* of the second kind.

$$E(k) := E\left(k, \frac{\pi}{2}\right) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi$$

— complete *elliptic integral* of the second kind.

$$\text{The length of the arc } (0 \leq \varphi \leq \theta) = aE\left(\frac{a^2 - b^2}{a^2}, \theta\right).$$

$$\text{The length of the ellipse} = 4aE\left(\frac{a^2 - b^2}{a^2}\right).$$

Except for the case $a = b$ (i.e., circles), such an integral cannot be expressed in terms of elementary functions.

(That's why we didn't learn this formula in schools!)

- Another expression of the elliptic integral of the second kind

Let us compute the arc length using the parametrisation:

$$(x, y(x)) = \left(x, b\sqrt{1 - \frac{x^2}{a^2}} \right). \quad (x \in [0, a \cos \theta])$$

The arc length = $aE(k, \theta)$

$$\begin{aligned} &= \int_0^{a \sin \theta} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= \int_0^{a \sin \theta} \sqrt{1 + \frac{b^2}{a^2} \frac{(x/a)^2}{1 - (x/a)^2}} dx \\ &= a \int_0^{\sin \theta} \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz. \quad (z = x/a) \end{aligned}$$

In particular,

$$E(k, \theta) = \int_0^{\sin \theta} \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz,$$

$$E(k) = \int_0^1 \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz.$$

Exercise:

(i) Find the arc length of other quadratic curves, i.e., of a parabola and a hyperbola. Which of them is expressed by an elliptic integral?

(ii) Express the arc length of the graph of $y = b \sin \frac{x}{a}$ in terms of the elliptic integral of the second kind. What arc correspond to $E(k)$?

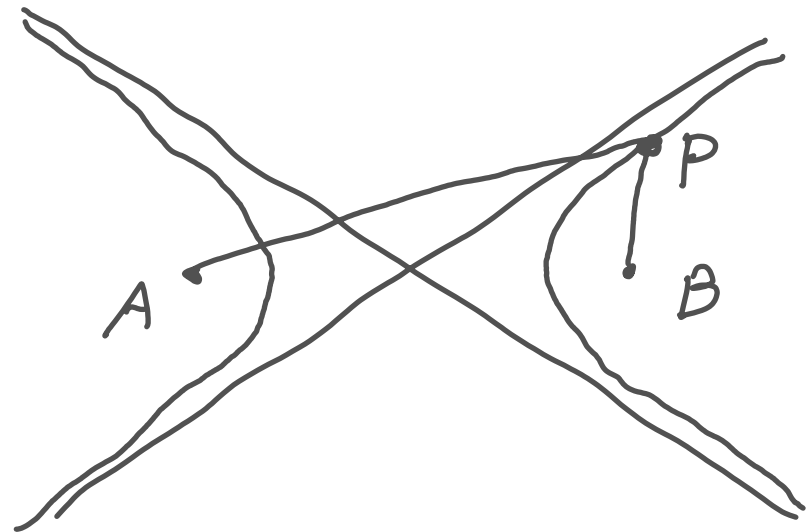
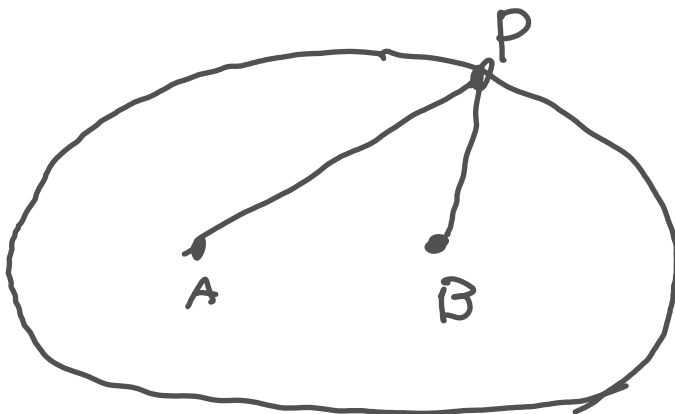
§1.2 Lemniscate and its arc length

Fix two points A and B on a plane and a positive number l .

$$\text{Ellipse} = \{P \mid PA + PB = l\}.$$

$$\text{Hyperbola} = \{P \mid PA - PB = \pm l\}.$$

What is the curve defined by an equation $PA \times PB = \text{constant} (= l^2)$?



$P = (x, y) = (r \cos \varphi, r \sin \varphi)$, $A = (-a, 0)$, $B = (a, 0)$:

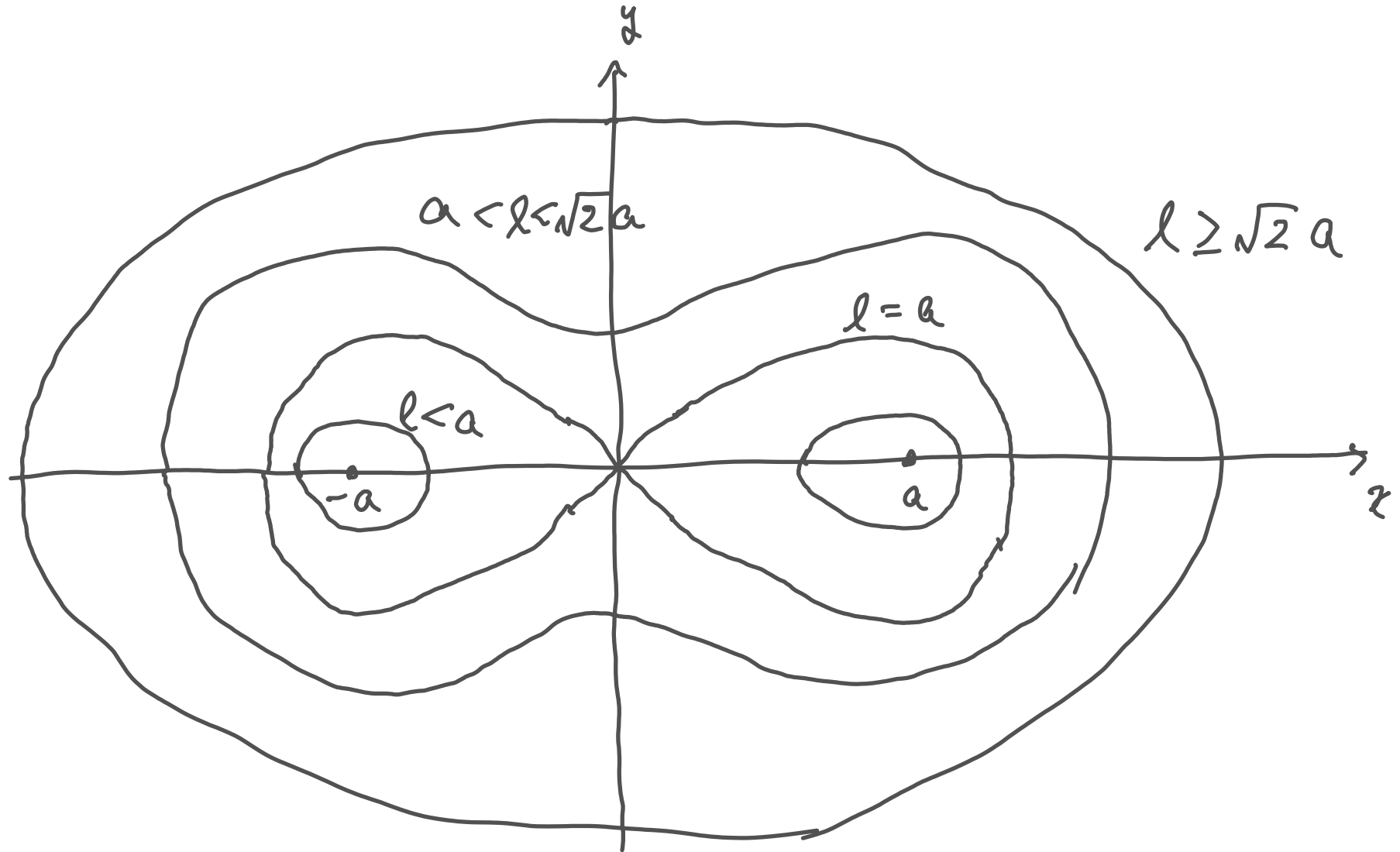
$$\begin{aligned} l^2 = PA \cdot PB &= \sqrt{(x+a)^2 + y^2} \sqrt{(x-a)^2 + y^2} \\ &= \sqrt{x^2 + y^2 + 2ax + a^2} \sqrt{x^2 + y^2 - 2ax + a^2} \\ &= \sqrt{r^2 + 2ar \cos \varphi + a^2} \sqrt{r^2 - 2ar \cos \varphi + a^2} \\ &= \sqrt{(r^2 + a^2)^2 - 4a^2 r^2 \cos^2 \varphi} \\ &= \sqrt{r^4 + a^4 - 2a^2 r^2 \cos 2\varphi}. \end{aligned}$$

By squaring, we obtain an quartic equation:

Cassini's oval : $r^4 + a^4 - 2a^2 r^2 \cos 2\varphi = l^4$.

The case $l = a$ is called the *lemniscate*.

Figures of Cassini's oval and the lemniscate:



Equations for the lemniscate:

In polar coordinates:

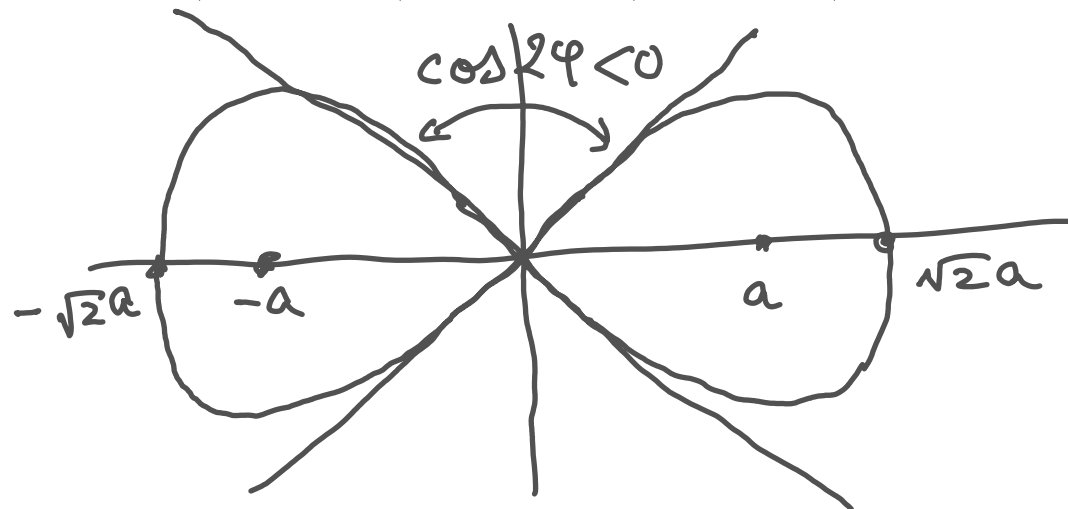
$$r^4 = 2a^2 r^2 \cos 2\varphi, \text{ i.e., } r^2 = 2a^2 \cos 2\varphi \text{ and } r = 0.$$

($\varphi \notin (\pi/4, 3\pi/4) \cup (5\pi/4, 7\pi/4)$, since $\cos 2\varphi$ should not be negative.)

In Cartesian coordinates:

$$r^2 = x^2 + y^2, \quad r^2 \cos 2\varphi = r^2 \cos^2 \varphi - r^2 \sin^2 \varphi = x^2 - y^2, \text{ hence}$$

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$



New parametrisation: $r = \sqrt{2}a \cos \psi$. (Note: $r^2 \leq 2a^2$, i.e., $r \leq \sqrt{2}a$.)

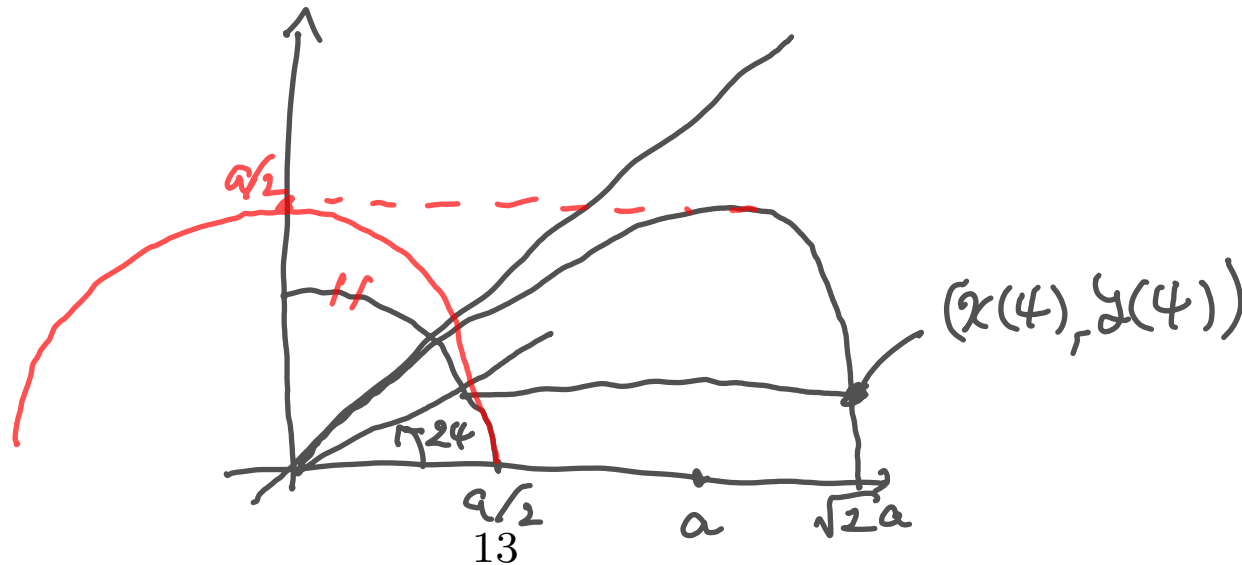
$$(x^2 + y^2)^2 = r^4 = 4a^4 \cos^4 \psi, \text{ hence, } x^2 - y^2 = 2a^2 \cos^4 \psi.$$

Together with $x^2 + y^2 = 2a^2 \cos^2 \psi$,

$$x^2 = a^2 \cos^2 \psi (1 + \cos^2 \psi), \quad y^2 = a^2 \cos^2 \psi (1 - \cos^2 \psi).$$

or, in the first quadrant ($x \geq 0, y \geq 0; 0 \leq \psi \leq \pi/4$),

$$x = \sqrt{2}a \cos \psi \sqrt{1 - \frac{1}{2} \sin^2 \psi}, \quad y = a \cos \psi \sin \psi = \frac{a}{2} \sin 2\psi.$$



Arc length of the lemniscate:

$$\frac{dx}{d\psi} = \sqrt{2}a \frac{\sin \psi}{\sqrt{1 - \frac{1}{2} \sin^2 \psi}} \left(-\frac{3}{2} + \sin^2 \psi \right),$$

$$\frac{dy}{d\psi} = a(1 - 2 \sin^2 \psi).$$

$$\implies \left(\frac{dx}{d\psi} \right)^2 + \left(\frac{dy}{d\psi} \right)^2 = \frac{a^2}{1 - \frac{1}{2} \sin^2 \psi}.$$

So, the arc length of the lemniscate is equal to

$$a \int_0^\varphi \frac{d\psi}{\sqrt{1 - \frac{1}{2} \sin^2 \psi}}.$$

$$F(k, \varphi) := \int_0^\varphi \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

— incomplete *elliptic integral* of the first kind.

$$K(k) := F\left(k, \frac{\pi}{2}\right) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

— complete *elliptic integral* of the first kind.

The length of the arc $(0 \leq \psi \leq \varphi) = aF\left(\frac{1}{\sqrt{2}}, \varphi\right)$.

The length of the lemniscate $= 4aK\left(\frac{1}{\sqrt{2}}\right)$.

- Another expression of the elliptic integral of the first kind

Change the integration variable from ψ to $z := \sin \psi$:

$$dz = \cos \psi d\psi = \sqrt{1 - z^2} d\psi$$

$$\implies F(k, \varphi) = \int_0^{\sin \varphi} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},$$

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$