Elliptic Functions

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4 April 2016

- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

  $$(\text{your final mark}) = \min\left\{\text{integer part of } \frac{3}{2}(\text{total points you get}), 10\right\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **11** – **13**: 18 April 2016.

11. (1 pt.) Let $\varphi(z)$ be a polynomial satisfying the conditions in the lecture (4 April 2016). Show the following:

(i) $\mathcal{R}_\varphi := \{(z,w) \in \mathbb{C}^2 \mid F_\varphi(z,w) := w^2 - \varphi(z) = 0\}$ is a non-singular algebraic curve over $\mathbb{C}$.

(ii) The 1-form $\omega := \dfrac{dz}{w}$ is holomorphic everywhere on $\mathcal{R}_\varphi$.

12. Show that the closure $\bar{\mathcal{R}}_\varphi$ of $\mathcal{R}_\varphi$ in $\mathbb{P}^2(\mathbb{C})$ (cf. lecture on 4 April 2016) is

(i) (1 pt.) *non-singular* if $\deg \varphi(z) = 3$.

(ii) (1 pt.) *singular* if $\deg \varphi(z) = 4$.

13. (i) (2 pt.) Show that the elliptic curve $\bar{\mathcal{R}}_\varphi$ ($\deg \varphi = 3$ or 4) is isomorphic to $\bar{\mathcal{R}}_\psi$, $\psi(z) = (1-z^2)(1-k^2z^2)$ for some $k \in \mathbb{C} \smallsetminus \{0, \pm 1\}$. Namely, construct a biholomorphic bijection $\Phi : \bar{\mathcal{R}}_\varphi \xrightarrow{\sim} \bar{\mathcal{R}}_\psi$.

(ii) (2 pt.) The same for $\psi(z) = z(1-z)(1-\lambda z)$, $\lambda \in \mathbb{C} \smallsetminus \{0, 1\}$.