Elliptic Functions

Complex Elliptic Integrals
§6.1 Complex elliptic integral of the first kind

Want: elliptic integrals \( \int_C R(x, \sqrt{\varphi(x)}) \, dx \) with complex variables.

\( C \): curve on the Riemann surface \( \mathcal{R} \) of \( \sqrt{\varphi(z)} \),
or its compactification \( \overline{\mathcal{R}} = \) the elliptic curve.

Let us begin with \( \int \frac{dz}{\sqrt{\varphi(z)}} \), the elliptic integral of the first kind.

\[ \omega_1 := \frac{dz}{\sqrt{\varphi(z)}} = \frac{dz}{w}. \]
We know that $\omega_1$ is holomorphic on

$$\mathcal{R} = \mathcal{R} \setminus \{\infty\} \ (\text{deg } \varphi = 3) \text{ or } \mathcal{R} = \mathcal{R} \setminus \{\infty_{\pm}\} \ (\text{deg } \varphi = 4).$$

How about the neighbourhood of $\infty_{\pm}$?

Assume $\text{deg } \varphi = 4$: $\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$.

(The case $\text{deg } \varphi = 3$ is similar.)

Recall:

- local parameter at $\infty_{\pm} = \xi = z^{-1}$.
- the equation of $\mathcal{R}$ in the neighbourhood of $\infty_{\pm}$:

  $$\eta^2 = a(1 - \alpha_0 \xi)(1 - \alpha_1 \xi)(1 - \alpha_2 \xi)(1 - \alpha_3 \xi),$$

  where $\eta = wz^{-2}$.
- $\infty_{\pm} = (\xi = 0, \eta = \pm \sqrt{a} \neq 0)$. 
Consequently,

- $d\xi = -z^{-2} \, dz$.
- $\omega_1 = \frac{dz}{w} = -\frac{d\xi}{\eta}$.
- $\eta(\xi) = \sqrt{a(1 - \alpha_0 \xi)(1 - \alpha_1 \xi)(1 - \alpha_2 \xi)(1 - \alpha_3 \xi)}$: holomorphic in $\xi$.
- $\eta(\xi) \neq 0$ at $\infty_{\pm}$.

$\Longrightarrow \omega_1$ is holomorphic at $\infty_{\pm}$.

**Conclusion:** $\omega_1$ is holomorphic everywhere on $\bar{R}$.

Moreover, $\omega_1 \neq 0$ everywhere on $\bar{R}$ ($\iff \omega_1 = \frac{dz}{w} = -\frac{d\xi}{\eta}$).

**Exercise:** Show that $\omega_1$ is holomorphic everywhere and nowhere-vanishing in the case $\deg \varphi = 3$. 


Fix $P_0 \in \overline{\mathcal{R}}$.

$\omega_1$ is holomorphic on $\overline{\mathcal{R}}$.

$$\implies F(P) := \int_{P_0 \to P} \omega_1 = \int_{C: \text{contour from } P_0 \text{ to } P} \omega_1$$

is "locally" well-defined and holomorphic in $P$.

Figure of $\overline{\mathcal{R}}$ and $C$:

"$F(P)$ is \textit{locally} well-defined" = does not change by small change of $C$.

Exactly speaking,

$$[C] - [C'] = 0 \text{ in } H_1(\overline{\mathcal{R}}, \mathbb{Z}) \implies \int_C \omega_1 = \int_{C'} \omega_1.$$
Figure: $[C] - [C'] = 0.$

Is $F(P)$ “globally” well-defined?

Need to know: How many “globally” different contours exist on $\bar{\mathcal{R}}$?

Answer from topology: $H_1(\bar{\mathcal{R}}, \mathbb{Z}) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$, which means:

for $\forall$ closed curve $C$ on $\bar{\mathcal{R}}$, $\exists! m, n \in \mathbb{Z}$, such that

$$[C] = m[A] + n[B] \text{ in } H_1(\bar{\mathcal{R}}, \mathbb{Z}).$$

Figure: $A$-cycle and $B$-cycle.
$C_0$, $C_1$: curves from $P_0$ to $P$.

$\implies [C_1] - [C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

\[
\int_{C_1} \omega_1 = \int_{C_0} \omega_1 + m \int_A \omega_1 + n \int_B \omega_1.
\]

We call

\[
\int_A \omega_1: A\text{-period of 1-form } \omega_1, \quad \int_B \omega_1: B\text{-period of 1-form } \omega_1.
\]

Let us compute $A$- and $B$-periods for the case $\varphi(z) = (1 - z^2)(1 - k^2 z^2)$, i.e.,

\[
\omega_1 = \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.
\]

For simplicity, assume $k \in \mathbb{R}$, $0 < k < 1$. 
Recall the construction of $\tilde{R}$:

Two $\mathbb{P}^1$’s are glued together along cuts between two pairs of roots of $\varphi(z)$.

$$\text{roots of } \varphi(z) = \pm 1, \pm k^{-1}.$$

Cut $\mathbb{P}^1$’s along $[-k^{-1}, -1]$ and $[1, k^{-1}]$ and glue.

(Figure of $A$- and $B$-cycles on $\mathbb{P}^1$’s)
Periods of $\omega_1$:

$$\int_A \omega_1 = 4 K(k), \quad \int_B \omega_1 = 2i K'(k),$$

where

- $K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$: complete elliptic integral of the first kind.

- $K'(k) := K(k')$, $k' := \sqrt{1 - k^2}$ (supplementary modulus).

Proof:

$$\int_A \omega_1 = \int_{-1}^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} + \int_1^{-1} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

(Note: $\pm$ because of branches.)

$$= 4 \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)} = 4 K(k).}$$
\[ \int_{B} \omega_{1} = \int_{1}^{1/k} dx + \sqrt{(1 - x^{2})(1 - k^{2}x^{2})} + \int_{1/k}^{1} -\sqrt{(1 - x^{2})(1 - k^{2}x^{2})} \]

\[ = 2 \int_{1}^{1/k} \frac{dx}{\sqrt{(1 - x^{2})(1 - k^{2}x^{2})}} = 2i \int_{1}^{1/k} \frac{dx}{\sqrt{(x^{2} - 1)(1 - k^{2}x^{2})}} \]

Change of the variable: \( x = \frac{1}{\sqrt{1 - k'^{2}t^{2}}} \), i.e., \( x^{2} = \frac{1}{1 - k'^{2}t^{2}} \),

\[ dx = \frac{k'^{2}t}{(1 - k'^{2}t^{2})^{3/2}} dt, \quad (x^{2} - 1)(1 - k^{2}x^{2}) = \frac{k'^{4}t^{2}(1 - t^{2})}{(1 - k'^{2}t^{2})^{2}}. \]

Hence,

\[ \int_{B} \omega_{1} = 2i \int_{0}^{1} \frac{dt}{\sqrt{(1 - t^{2})(1 - k'^{2}t^{2})}} = 2i K(k') = 2i K'(k). \]
Remark:

- Signs of $\sqrt{\cdot}$ should be chosen carefully.
- For general $k \in \mathbb{C}$, the results are the same (analytic continuation).

Recall:

"$A$-period of $\frac{dz}{\sqrt{1-z^2}} = 2\pi = \text{period of } \sin(u)$.”

Correspondingly,

$A$-period of $\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 4K(k) = \text{period of } \text{sn}(u)$!

What is the role of the $B$-period $2iK'(k)$ for $\text{sn}(u)$?

$\rightarrow$ Another period of $\text{sn}(u)$, i.e., $\text{sn}(u)$ is doubly-periodic!

Details will be discussed later...
§6.2 Complex elliptic integral of the second kind

\[ \int \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} \, dz = \int \frac{1 - k^2 z^2}{\sqrt{\varphi(z)}} \, dz, \quad \varphi(z) = (1 - z^2)(1 - k^2 z^2). \]

Corresponding Riemann surface = \( R = \{(z, w) \mid w^2 = \varphi(z)\} \) as before.

The compactification = \( \tilde{R} \): elliptic curve.

\[ \omega_2 := \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} \, dz = \frac{1 - k^2 z^2}{\sqrt{\varphi(z)}} \, dz = \frac{1 - k^2 z^2}{w} \, dz \]

is holomorphic on \( R \) as \( \omega_1 \). (In particular, at \( z = \pm 1, \pm k^{-1} \).)

How about at \( \{\infty_\pm\} = \tilde{R} \setminus R \)?
Local coordinate at ±∞: \( \xi = z^{-1} \).

\[
\omega_2 = \sqrt{\frac{1 - k^2 \xi^{-2}}{1 - \xi^{-2}}} \, d(\xi^{-1}) = \sqrt{\frac{\xi^2 - k^2}{\xi^2 - 1}} \cdot (-\xi^{-2}) \, d\xi
\]

\[
= -\xi^{-2} (\pm k + O(\xi^2)) \, d\xi = \left( \mp \frac{k}{\xi^2} + (\text{holomorphic at } \xi = 0) \right) \, d\xi.
\]

\[\implies \omega_2 \text{ has double poles at } \infty_{\pm} \text{ without residues: } \text{Res}_{\infty_{\pm}} \omega_2 = 0.\]

\[\implies G(P) := \int_{P_0 \to P} \omega_2 = \int_{C: \text{ contour from } P_0 \text{ to } P} \omega_2 \text{ is}
\]

- locally well-defined.
- holomorphic in \( P \) except at \( \infty_{\pm} \).
- has a simple pole at \( \infty_{\pm} \): \( G(P) = \pm \frac{k}{\xi} + (\text{holomorphic at } \xi = 0) \).
Global multi-valuedness: same as the case of $\omega_1$.

$C_0, C_1$: curves from $P_0$ to $P$.

$\implies [C_1] - [C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

\[
\int_{C_1} \omega_2 = \int_{C_1} \omega_2 + m \int_A \omega_2 + n \int_B \omega_2.
\]

$\int_A \omega_2$: $A$-period of $\omega_2$, $\int_B \omega_2$: $B$-period of $\omega_2$.

Exercise: Express $A$- and $B$-periods of $\omega_2$ in terms of the complete elliptic integral of the second kind.
§6.3 Complex elliptic integral of the third kind

\[ \int \frac{dz}{(z^2 - a^2) \sqrt{\varphi(z)}}, \quad \varphi(z) = (1 - z^2)(1 - k^2 z^2). \]

\[ \omega_3 := \frac{dz}{(z^2 - a^2) \sqrt{\varphi(z)}} = \frac{dz}{(z^2 - a^2)w} \]

is holomorphic on the elliptic curve (including \( \infty_{\pm} \)) except at four points:

\[ (z, w) = (\pm a, \pm \sqrt{(1 - a^2)(1 - k^2 a^2)}). \]

These are simple poles.

Exercise: (i) Check these facts. (ii) Compute the residues at poles.
\[ H(P) := \int_{P_0 \to P} \omega_3 \]

multi-valued in the neighbourhood of simple poles because of the residue.
(And, of course, globally multi-valued.)
\[ \implies H(P) \text{ is a very complicated function.} \]
Remark:

A meromorphic 1-form $\omega$ on a Riemann surface is called an *Abelian differential*. It is

- *of the first kind*, when $\omega$ is holomorphic everywhere.
- *of the second kind*, when the residue is zero at any pole.
- *of the third kind*, otherwise.

$\implies \omega_1$: the first kind, $\omega_2$: the second kind, $\omega_3$: the third kind.

(There are several different definitions; e.g.,

- “‘the third kind’ has only simple poles’,
- “‘the second kind’ has only one pole of order $\geq 2’$, etc.)