

Elliptic Functions

Complex Jacobian elliptic functions

§11.1 Definition of Jacobian elliptic functions in terms of θ .

Recall: Jacobi's $\text{sn}(u, k)$ was defined as the inverse function of

$$u = \int_0^{\text{sn}(u, k)} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Exercise of the last section: \forall elliptic functions = rational function of θ .

Question: *How can Jacobian functions be defined by θ ?*

Answer: $v := \frac{u}{\pi\theta_{00}^2}, k := \frac{\theta_{10}^2}{\theta_{00}^2},$

$$\text{sn}(u, k) := -\frac{\theta_{00} \theta_{11}(v)}{\theta_{10} \theta_{01}(v)}, \quad \text{cn}(u, k) := \frac{\theta_{01} \theta_{10}(v)}{\theta_{10} \theta_{01}(v)}, \quad \text{dn}(u, k) := \frac{\theta_{01} \theta_{00}(v)}{\theta_{00} \theta_{01}(v)}.$$

- $\operatorname{sn}(u)$ defined above has *two periods*: $2\pi \theta_{00}^2$, $\pi \theta_{00}^2 \tau$.

Exercise: Check this. (Hint: quasi-periodicity of θ -functions.)

- $\operatorname{sn}(u)$ is meromorphic. $\iff \theta$'s are entire.

$\implies \operatorname{sn}(u)$ is an elliptic function.

Similarly,

- $\operatorname{cn}(u)$ is an elliptic function with periods $2\pi \theta_{00}^2$, $\pi \theta_{00}^2(1 + \tau)$.
- $\operatorname{dn}(u)$ is an elliptic function with periods $\pi \theta_{00}^2$, $2\pi \theta_{00}^2 \tau$.

Let us check that they coincide what we defined before (on \mathbb{R}).

- $\text{sn}(0) = 0, \text{cn}(0) = \text{dn}(0) = 1 \Leftrightarrow \theta_{11}(0) = 0$ & definitions.
- sn : odd, cn, dn : even $\Leftrightarrow \theta_{11}(u)$: odd, $\theta_{kl}(u)$: even ($(k, l) \neq (1, 1)$).
- $\text{sn}^2 u + \text{cn}^2 u = 1, k^2 \text{sn}^2 u + \text{dn}^2 u = 1$.

Proof:

$$\text{sn}^2 u + \text{cn}^2 u = \frac{\theta_{00}^2 \theta_{11}(v)^2 + \theta_{01}^2 \theta_{10}(v)^2}{\theta_{10}^2 \theta_{01}(v)^2}.$$

Recall the addition formula (A1):

$$\begin{aligned} \theta_{00}(x+u) \theta_{00}(x-u) \theta_{00}^2 &= \theta_{01}(x)^2 \theta_{01}(u)^2 + \theta_{10}(x)^2 \theta_{10}(u)^2 \\ &= \theta_{00}(x)^2 \theta_{00}(u)^2 + \theta_{11}(x)^2 \theta_{11}(u)^2. \end{aligned}$$

$$x = v, u = \frac{1+\tau}{2} \implies \theta_{11}(v)^2 \theta_{00}^2 = \theta_{01}(v)^2 \theta_{10}^2 - \theta_{10}(v)^2 \theta_{01}^2.$$

$$\implies \text{sn}^2 u + \text{cn}^2 u = 1.$$

$$x = v, u = \frac{1}{2} \text{ in (A1): } \theta_{01}(v)^2 \theta_{00}^2 = \theta_{00}(v)^2 \theta_{01}^2 + \theta_{11}(v)^2 \theta_{10}^2.$$

$$\implies \frac{\theta_{10}^4 \theta_{00}^2 \theta_{11}(v)^2}{\theta_{00}^4 \theta_{10}^2 \theta_{01}(v)^2} + \frac{\theta_{01}^2 \theta_{11}(v)^2}{\theta_{00}^2 \theta_{01}(v)^2} = 1, \text{ i.e., } k^2 \operatorname{sn}^2(u) + \operatorname{dn}^2(u) = 1.$$

□

- $\frac{d}{du} \operatorname{sn}(u) = \operatorname{cn}(u) \operatorname{dn}(u).$

Proof:

Chain rule & $v = \frac{u}{\pi \theta_{00}^2}$

$$\begin{aligned} \implies \frac{d}{du} \operatorname{sn}(u) &= \frac{dv}{du} \frac{d}{dv} \left(-\frac{\theta_{00}}{\theta_{10}} \frac{\theta_{11}(v)}{\theta_{01}(v)} \right) \\ &= -\frac{1}{\pi \theta_{00} \theta_{10}} \frac{\theta'_{11}(v) \theta_{01}(v) - \theta_{11}(v) \theta'_{01}(v)}{\theta_{01}(v)^2}. \end{aligned}$$

Recall the addition formula (A3):

$$\theta_{11}(x+u)\theta_{01}(x-u)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(u)\theta_{11}(u) + \theta_{01}(x)\theta_{11}(x)\theta_{00}(u)\theta_{10}(u).$$

Expand around $u = 0$ and take the coefficients of u^1 :

$$(\theta'_{11}(x)\theta_{01}(x) - \theta_{11}(x)\theta'_{01}(x))\theta_{00}\theta_{10} = \theta_{00}(x)\theta_{10}(x)\theta_{01}\theta'_{11}.$$

Substitute this into the previous equation ($x \mapsto v$):

$$\begin{aligned} \frac{d}{du} \operatorname{sn}(u) &= -\frac{1}{\pi\theta_{00}\theta_{10}} \frac{\theta_{00}(v)\theta_{10}(v)\theta_{01}\theta'_{11}}{\theta_{00}\theta_{10}\theta_{01}(v)^2} \\ &= \frac{\theta_{01}^2}{\theta_{00}\theta_{10}} \frac{\theta_{00}(v)\theta_{10}(v)}{\theta_{01}(v)^2} \quad (\text{Jacobi's derivative formula}) \\ &= \operatorname{cn}(u)\operatorname{dn}(u). \end{aligned}$$

□

As we have seen in the real case, the above formulae lead to

$$\frac{d}{du} \operatorname{sn}(u) = \sqrt{(1 - \operatorname{sn}^2(u))(1 - k^2 \operatorname{sn}^2(u))}.$$
$$\implies u = \int_0^{\operatorname{sn}(u)} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

Consistent with the previous definition.

§11.2 Properties of $\operatorname{sn}(u, k)$.

What we know about Jacobi's functions $/\mathbb{R}$:

- periodicity (e.g., period of $\operatorname{sn} = 4K(k)$).
- limits (e.g., $\operatorname{sn} \rightarrow \sin$ when $k \rightarrow 0$).
- addition formulae.

can be checked on the basis of the definition by θ .

Because of the lack of time, we prove only the addition formula for sn .

Proofs of other properties are only sketched.

- Addition formula of sn.

Recall addition formulae (A3) & (A2) of θ 's:

$$\begin{aligned} \theta_{11}(x+y)\theta_{01}(x-y)\theta_{10}\theta_{00} &= \theta_{00}(x)\theta_{10}(x)\theta_{01}(y)\theta_{11}(y) \\ &\quad + \theta_{01}(x)\theta_{11}(x)\theta_{00}(y)\theta_{10}(y), \end{aligned}$$

$$\theta_{01}(x+y)\theta_{01}(x-y)\theta_{01}^2 = \theta_{01}(x)^2\theta_{01}(y)^2 - \theta_{11}(x)^2\theta_{11}(y)^2.$$

Set $u = \pi \theta_{00}^2 x$, $v = \pi \theta_{00}^2 y$:

$$-(\text{ratio of LHS's}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = -\frac{\theta_{00} \theta_{11}(x+y)}{\theta_{10} \theta_{01}(x+y)} = \text{sn}(u+v).$$

$$-(\text{ratio of RHS's}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = \frac{\text{sn}(u) \text{cn}(v) \text{dn}(v) + \text{sn}(v) \text{cn}(u) \text{dn}(u)}{1 - k^2 \text{sn}(u)^2 \text{sn}(v)^2},$$

as was proved before. □

- Limits $k \rightarrow 0, k \rightarrow 1$.

$$\underline{k = k(\tau) \rightarrow 0} \iff \tau \rightarrow i\infty.$$

In this limit: $\theta_{11}(u, \tau) \sim \sin u$, $\theta_{01}(u, \tau) \sim 1$, etc. $\implies \operatorname{sn}(u, k) \rightarrow \sin(u)$.

$$\underline{k \rightarrow 1} \iff k' \rightarrow 0 \quad (k' := \sqrt{1 - k^2}).$$

Modular properties: relations of $\theta_{kl}(u, \tau)$ and $\theta_{k'l'}\left(\frac{u}{\tau}, -\frac{1}{\tau}\right)$.

$$\implies \begin{cases} k' = k'(\tau) = k\left(-\frac{1}{\tau}\right), \\ \operatorname{sn}(iu, k) = \frac{i \operatorname{sn}(u, k')}{\operatorname{cn}(u, k')} \text{ etc.} \end{cases}$$

$$\implies \lim_{k \rightarrow 1} \operatorname{sn}(u, k) = \tanh(u) \text{ etc.}$$

□

- Periodicity.

Recall: for $k = \frac{\theta_{10}^2}{\theta_{00}^2}$,

$$x = \operatorname{sn}(u) = -\frac{\theta_{00}}{\theta_{10}} \frac{\theta_{11}\left(\frac{u}{\pi\theta_{00}^2}, \tau\right)}{\theta_{01}\left(\frac{u}{\pi\theta_{00}^2}, \tau\right)} \xleftrightarrow{\text{inverse}} u = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

A- & *B*-periods of the elliptic integral (RHS):

$$4K(k) = 4 \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

$$2iK'(k) = 2 \int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Periods of $\operatorname{sn}(u)$ (defined by θ -functions; LHS): $2\pi\theta_{00}^2, \pi\theta_{00}^2\tau$.

$$\implies 4\mathbb{Z}K(k) + 2\mathbb{Z}iK'(k) = 2\mathbb{Z}\pi\theta_{00}^2 + \mathbb{Z}\pi\theta_{00}^2\tau.$$

Or, equivalently, $\exists m_1, m_2, n_1, n_2 \in \mathbb{Z}$,

$$4K(k) = 2m_1\pi\theta_{00}^2 + n_1\pi\theta_{00}^2\tau, \quad 2iK'(k) = 2m_2\pi\theta_{00}^2 + n_2\pi\theta_{00}^2\tau.$$

Theorem:

$$K(k) = \frac{\pi}{2}\theta_{00}^2, \quad K'(k) = \frac{\pi}{2i}\theta_{00}^2\tau.$$

Proof is not difficult but lengthy.

We omit it here because of the lack of time. □