Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The solution of this sheet is not reflected to the evaluation. Therefore no deadline is set.

23. (1 pt.) Show that an entire function $f(u)$ satisfying $f(u+1) = f(u)$, $f(u+\tau) = e^b f(u)$ ($b \in \mathbb{C}$) is proportional to $e^{2\pi i n u}$ for some $n \in \mathbb{Z}$. (Hint: Prove that, if a coefficient a_n in the Fourier expansion $f(u) = \sum a_n e^{2\pi i n u}$ is not zero, then $\operatorname{Re} b = 2\pi n \operatorname{Im}\tau$. Therefore n is uniquely determined by b.)

24. (1 pt.) Fix $k \in \mathbb{Z}_{>0}$ and $b \in \mathbb{C}$. Show that the dimension of the linear space $\Theta_{k,b}$ of all entire functions satisfying $f(u+1) = f(u)$ and $f(u+\tau) = e^{-2\pi i k u + b} f(u)$ is equal to k. Find a basis of $\Theta_{k,b}$ expressed by theta functions.

25. (1 pt.) Let a_i and b_i ($i = 1, \dots, N$) be complex numbers satisfying $\sum a_i = \sum b_i$. Show that a function of the form

$$f(u) = c\frac{\theta_{11}(u-a_1)\cdots\theta_{11}(u-a_N)}{\theta_{11}(u-b_1)\cdots\theta_{11}(u-b_N)}$$

($c \in \mathbb{C}$) is an elliptic function with periods 1 and τ and that conversely any elliptic function with periods 1 and τ has this form.

26. (1 pt.) Prove the following formulae (*Landen's transformation*):

$$\theta_{01}(2u, 2\tau) = \frac{\theta_{01}(0, 2\tau)}{\theta_{01}(0,\tau)\theta_{00}(0,\tau)}\theta_{00}(u,\tau)\theta_{01}(u,\tau),$$
$$\theta_{11}(2u, 2\tau) = \frac{\theta_{01}(0, 2\tau)}{\theta_{01}(0,\tau)\theta_{00}(0,\tau)}\theta_{10}(u,\tau)\theta_{11}(u,\tau).$$

(Hint: Compare the zeros and quasi-periodicity of both sides. If the zeros and quasi-periodicity are the same, the ratio of them should be a holomorphic elliptic function. Apply Liouville's theorem. The constant in the first equation can be found easily. The second equation is derived from the first by shifting $u \mapsto u+\tau/2$.)