# $C^*$-ALGEBRAS AND COMPACT QUANTUM GROUPS

(Spring 2017)

A. Yu. Pirkovskii

The $C^*$-algebra theory is an algebraic branch of functional analysis. It appeared in the 1940ies in the foundational papers of I. M. Gelfand and M. A. Naimark, and has evolved into an extremely deep, multi-branch mathematical discipline since then. A $C^*$-algebra is a $\mathbb{C}$-algebra equipped with a norm and an involution satisfying some compatibility axioms. The basic examples are the algebra $C(X)$ of continuous functions on a compact topological space $X$ and the algebra $\mathscr{B}(H)$ of bounded linear operators on a Hilbert space $H$. These examples are "universal" due to the following Gelfand-Naimark Theorems: (1) each commutative $C^*$-algebra with identity is isomorphic to $C(X)$ for some $X$, and (2) each $C^*$-algebra can be isometrically embedded into $\mathscr{B}(H)$ for some $H$. The 1st Gelfand-Naimark Theorem (statement (1) above) lies at the foundation of noncommutative geometry (à la A. Connes) and the theory of compact quantum groups.

The theory of compact quantum groups was created mostly by S. L. Woronowicz in the 1980ies-1990ies. Loosely speaking, a compact quantum group is a "deformation" of the algebra of continuous functions on a compact topological group. Thus, according to the well-known saying, quantum groups are "neither quantum, nor groups". In Woronowicz's theory, a compact quantum group is a $C^*$-algebra endowed with an additional structure (comultiplication) satisfying some natural axioms[1]. Many classical results of the theory of compact groups (the existence and uniqueness of the Haar measure, the complete reducibility of unitary representations, the Peter-Weyl theorem, the Tannaka-Krein duality, etc.) have natural "quantum" analogs. The theory of compact quantum groups is only a part of a much more general (and much more difficult) theory of locally compact quantum groups developed by J. Kustermans and S. Vaes in the early 2000ies. Nowadays, this is one of the most popular and actively developing fields of operator algebra theory.

**Prerequisites.** The Lebesgue integration theory and the basics of functional analysis. Some knowledge of the representation theory of compact (or at least finite) groups will also be helpful.

## Syllabus

1. **$C^*$-ALGEBRAS.** Basic definitions and examples. Commutative $C^*$-algebras and the 1st Gelfand-Naimark Theorem. The functional calculus and positive elements in $C^*$-algebras. Representations of $C^*$-algebras. Positive functionals and the GNS construction. The 2nd Gelfand-Naimark Theorem. Some constructions: multiplier algebras, tensor products of $C^*$-algebras, $C^*$-envelopes. Basic facts on Hilbert $C^*$-modules.

2. **COMPACT QUANTUM GROUPS.** Definitions and examples (quantum $SU(n)$, quantum $SO(n)$, free unitary and free orthogonal quantum groups). Commutative compact quantum groups. The Haar state. Unitary corepresentations. Decomposing into irreducibles. The orthogonality relations. The Hopf subalgebra of matrix elements of finite-dimensional corepresentations. Remarks on the pairing with quantized enveloping algebras. The Tannaka-Krein duality.

[1] The $C^*$-algebra approach to quantum groups is closely related to the more popular algebraic approach via duality, real forms and $C^*$-completions, but in general there is no 1–1 correspondence between them.