Representations of affine and vertex operator algebras

Homework 1

1. Consider a representation of the Witt algebra with a basis v_k, $k \in \mathbb{Z}$:

$$L_n(v_k) = (k - \alpha - \beta(n+1))v_{k-n}.$$

Express the action of the operators L_n in terms of the matrix units $E_{i,j}$ and compute the value of the japanese cocycle on the pairs L_n, L_m.

2. Using the previous problem, construct the structure of the module of the Virasoro algebra on the space $F^{(m)}$ via the $\widehat{\mathfrak{a}}_\infty$ action on $F^{(m)}$. Find the highest weight of this representation, i.e. the eigenvalues of the operators c and L_0 on the vacuum vector $|m\rangle$.

3. Let B_μ be the Fock module of the Heisenberg algebra ($\hbar$ acts as the identity operator and a_0 acts via multiplication by μ). Consider the operators L_n:

$$L_0 = (\mu^2 + \lambda^2)/2 + \sum_{j>0} a_{-j}a_j,$$

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n} + i\lambda n a_n, \ n \neq 0.$$

Prove that the operators L_n satisfy the Virasoro algebra relations and find the central charge.

4. Prove Proposition 3.7 from the book Kac, Raina. Bombay lectures on Highest-weight representations of infinite-dimensional Lie algebras.

5. Prove that any one-dimensional central extension of a simple Lie algebra is trivial.