Representations of affine and vertex operator algebras

Exam

1. Fix $N, k > 0$ and consider the vector

$$w_{N,k} = v_N \wedge v_{N-1} \wedge \cdots \wedge v_{N-k+1} \wedge v_{-k} \wedge v_{-k-1} \wedge \ldots$$

in the space of semi-infinite forms $F^{(0)}$. Compute the q-dimension of the space $\mathbb{C}[\Lambda_1, \Lambda_2, \ldots] w_{N,k}$.

2. Prove that all the fundamental modules $F_n^{(m)}$ of $\widehat{\mathfrak{sl}}_n'$ can be extended to the $\widehat{\mathfrak{sl}}_n$ modules.

3. Prove that the characters of the fundamental $\widehat{\mathfrak{sl}}_2$ modules are given by the formulas

$$\mathrm{ch}_q L(\omega_0) = \prod_{i=1}^{\infty} (1-q^i)^{-1} \sum_{n\in\mathbb{Z}} e^{2n\omega} q^{n^2},$$

$$\mathrm{ch}_q L(\omega_1) = \prod_{i=1}^{\infty} (1-q^i)^{-1} \sum_{n\in\mathbb{Z}} e^{(2n+1)\omega} q^{n(n+1)}.$$

4. Give a closed formula for the q-dimension with respect to the principal grading of a fundamental representation of $\widehat{\mathfrak{sl}}_n$.

5. Let $L(\omega_0)$ be the basic fundamental $\widehat{\mathfrak{sl}}_2$ module. Let

$$f(z) = \sum_{n\in\mathbb{Z}} z^{-n} (f \otimes t^n).$$

Prove that $f(z)^2 |0\rangle = 0$ in $L(\omega_0)$.

6*. Let $\mathfrak{g} = \mathfrak{sl}_n$ and let J_a, J^a, $a = 1, \ldots, \dim \mathfrak{g}$ be dual bases of $\mathfrak{g}$ with respect to the Killing form. Fix a number $k \neq -n$ and consider the Segal-Sugawara operators

$$L_n = \frac{1}{2(k+n)} \sum_{a=1}^{\dim \mathfrak{g}} \left(\sum_{m<0} (J_a \otimes t^m)(J^a \otimes t^{n-m}) + \sum_{m\geq 0} (J^a \otimes t^{n-m})(J_a \otimes t^m) \right).$$

Prove that the operators L_n acting on a level k representation of $\mathfrak{g}$ (if well-defined) satisfy Virasoro algebra relations. Compute the central charge of the corresponding representation of the Virasoro algebra.