Mathematical models of economic systems

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People are not logical. They are *psycho*logical.

Anonymous

Introduction

Decision theory develops an economic model of a man.

Basic elements:

- Decision maker
- Set of alternatives (actions)
- States of the world
- Outcomes

Example 1

	Sunny	Rainy
Take an umbrella	3	2
Go without umbrella	4	1

Related disciplines

- Microeconomics; macroeconomics
- Finance
- Behavioral economics; experiments
- Neuroeconomics; neuromarketing
- Decision making among animals
- Game theory
- Social choice theory

Decision problem

Choose $x \in A \subseteq X$ to maximise $V(x)$,

- objects of choice: $x \in X$,
- objective function: $V(x)$,
- feasible choices: $A \subseteq X$.

Objects of choice, set X

- Consumption bundle
- Production plan
- Portfolios of K assets
- Number of children
- To cheat or not to cheat

What is x?

- A cold beer on a hot evening three days from now in the company of friends in an Islamic society that prohibits the consumption of alcohol.

- Time, social interactions, culture, tastes.
- All these things can be modeled by decision theory.

Evaluation of outcomes

- Utility function
- Profit function
- Expected utility function

Two approaches

Descriptive theory:

– How do people evaluate outcomes?

– What do people actually choose?

Normative theory:

– How should one decide?

– What advice should one give a decision maker?

Decision theory under certainty

Preferences as

- a reason of behavior and
- nothing more than a complete *description* of consistent behavior.

Decision theory under certainty

Definition: A *choice function* for a finite set X is a function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $c(A) \subseteq A$.

- A preference relation $\succsim$ induces a choice set:

$$c_{\succsim}(A) := \{x \in A \mid x \succsim y \text{ for all } y \in A\}.$$

Note: if $\succsim$ is acyclic then $c(A, \succsim) \neq \varnothing$.

Decisions under certainty

The preference order $\succeq$ on X expresses the relation

"at least as good as".

With every (weak) preference order one can associate

(i) a *strict preference relation* $\succ$ on X defined by

$$x \succ y \quad \Leftrightarrow \quad x \succeq y \ \textbf{and not} \ \ y \succeq x,$$

(ii) an *indifference relation* $\sim$ on X defined by

$$x \sim y \quad \Leftrightarrow \quad x \succeq y \ \textbf{and} \ \ y \succeq x.$$

Revealed preferences

- Weak Axiom of Revealed Preferences

If *x* is ever chosen when *y* is available, then there is no set of alternatives for which *y* chosen and *x* is not.

Rational preference relation

Axiom A1: *Completeness*

For all $x, y \in X$,

either $x \succeq y$ or $y \succeq x$.

Axiom A2: *Transitivity*

For all $x, y, z \in X$,

$x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

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Representation

Theorem

The following statements are equivalent:

(i) The preference order $\succeq$ on X satisfies Axioms **A1, A2,...**

(ii) There exists a function $V : X \to \mathbb{R}$ such that

$$x \succeq y \Leftrightarrow V(x) \geq V(y) \text{ for all } x, y \in X.$$

The function V is unique up to a monotone transformation.

Representation

- Every function $V : X \to \mathbb{R}$ induces an order on X.

  For all $x, y \in X$, define

  $$x \succeq y \quad \Leftrightarrow \quad V(x) \geq V(y).$$

- When can a preference order $\succsim$ be represented by a real-valued function, $V : X \to \mathbb{R}$?

Axioms of preference order

Consider a set X and a preference order $\succeq$ on X.

Axiom A1: *Completeness*

For all $x, y \in X$,

either $x \succeq y$ or $y \succeq x$.

Axiom A2: *Transitivity*

For all $x, y, z \in X$,

$x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

Representation theorem

Because of a preference order $\succeq$ satisfies A1, A2, it is a weak order.

Proposition: If X is a finite set, then the following statements are equivalent:

(i) The preference order $\succsim$ on X satisfies Axioms **A1** and **A2**.

(ii) There exists a utility function $V : X \to \mathbb{R}$ which represents the preference order $\succsim$.

Proof

1. $\Longrightarrow$ Let

$$V(x) := \#\{y \in X \mid x \succsim y\}.$$

V is a representation:

(a) If $x \sim x'$, then $\{y \in X \mid x \succsim y\} = \{y \in X \mid x' \succsim y\}$ and, hence, $V(x) = V(x')$.

(b) If $x \succ x'$, then $\{y \in X \mid x' \succsim y\} \subset \{y \in X \mid x \succsim y\}$ and, hence, $V(x) > V(x')$.

2. $\Longleftarrow$ V satisfies Axioms **A1** and **A2**. ■

Uncountable set

Axiom A3a: *Separability* ($\succeq$*-order density*)
There exists a countable set $Z \subseteq X$
which is $\succeq$*-order-dense*,
i.e., for all $x, y \in X \backslash Z$ such that $x \succ y$, there exists a $z \in Z$ such that

$$x \succeq z \succeq y.$$

Theorem *(Cantor 1915).*

The following statements are equivalent:

(i) The preference order $\succeq$ on X satisfies Axioms **A1, A2**, and **A3a.**

(ii) There exists a function $V : X \to \mathbb{R}$ such that $x \succeq y \Leftrightarrow V(x) \geq V(y)$ for all $x, y \in X$.

Proof.

1. $\Longrightarrow$ Forming equivalence classes for all $x \sim x'$,
w.l.o.g, assume that $x \succ y$ for all $x, y, \in \widehat{X}$.

Consider the subsets of the countable set $Z = \{z_1, z_2, z_3, \ldots\}$:

$$W(x|Z) \ : = \{z_i \in Z | \ x \succsim z_i\} \subset Z,$$
$$B(x|Z) \ : = \{z_i \in Z | \ z_i \succsim x\} \subset Z.$$

The sets $W(x|Z)$ and $B(x|Z)$ are also countable.

Let

$$V(x) := \sum_{i \in \{i | z_i \in W(x|Z)\}} \frac{1}{2^i} - \sum_{i \in \{i | z_i \in B(x|Z)\}} \frac{1}{2^i}$$

V is a representation:

Let $x \succ y$, then

$$W(x|Z) \supseteq W(y|Z)$$

and

$$B(x|Z) \subseteq B(y|Z).$$

Hence, for $x \succ y$,

$$
\begin{aligned}
& V(x) - V(y) \\
= & \left(\sum_{i \in \{i | z_i \in W(x|Z)\}} \frac{1}{2^i} - \sum_{i \in \{i | z_i \in B(x|Z)\}} \frac{1}{2^i} \right) \\
& - \left(\sum_{i \in \{i | z_i \in W(y|Z)\}} \frac{1}{2^i} - \sum_{i \in \{i | z_i \in B(y|Z)\}} \frac{1}{2^i} \right) \\
= & \left(\sum_{i \in \{i | z_i \in W(x|Z)\}} \frac{1}{2^i} - \sum_{i \in \{i | z_i \in W(y|Z)\}} \frac{1}{2^i} \right) \\
& + \left(\sum_{i \in \{i | z_i \in B(y|Z)\}} \frac{1}{2^i} - \sum_{i \in \{i | z_i \in B(x|Z)\}} \frac{1}{2^i} \right) \\
= & \left(\sum_{i \in \{i | x \succsim z_i \succsim y\}} \frac{1}{2^i} \right) + \left(\sum_{i \in \{i | x \succsim z_i \succsim y\}} \frac{1}{2^i} \right) > 0,
\end{aligned}
$$

since, by Axiom **3a,** there must be z_i such that

- either $x \succsim z_i \succ y$,
- or $x \succ z_i \succsim y$,
- or $x \succ z_i \succ y$.

2. $\Longleftarrow$ V satisfies Axioms **A1**, **A2** (easy)

and **A3** (difficult). ■

Uncountable sets X and continuous utility functions

Let X be a subset of a finite Euclidean space, $X \subseteq \mathbb{R}^n$.

For any $x \in X$, consider

- the set of all $y \in X$ which are better or equal than x, i.e.,
$$B(x) := \{y \in X \mid y \succeq x\}$$
and
- the set of all $y \in X$ which are worse or equal than x, i.e.,
$$W(x) := \{y \in X \mid y \preceq x\}.$$

Continuity

Axiom A3b: *Continuity*

For all $x \in X$,

$B(x)$ and $W(x)$ are *closed sets* in X.

Remark: *Continuity* in Axiom **3b** is equivalent to the familiar notion of continuity:

(i) $x^n \in B(x)$ for all n and

(ii) $x^n \longrightarrow x^0$

implies $x^0 \in B(x)$.

Representation theorem

Theorem *(Debreu 1952)*
Suppose X is a subset of a finite Euclidean space, $X \subseteq \mathbb{R}^n$..

The following statements are equivalent:

(i) The preference order $\succeq$ on X satisfies Axioms **A1, A2**, and **A3b.**

(ii) There exists a continuous function $V : X \to \mathbb{R}$ such that $x \succeq y \Leftrightarrow V(x) \geq V(y)$ for all $x, y \in X$.

Uniqueness

A function $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing if for all $a, b \in \mathbb{R}$, $a \neq b$,

$$a > b \Longrightarrow \phi(a) > \phi(b).$$

- Any function V representing a preference order $\succeq$ is only *unique up to positive monotone transformation*.

Uniqueness

Proof.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a *strictly increasing funktion*,

then $W = \phi \circ V$ is also a function representing $\succsim$.

$$x \succsim y \iff V(x) \geq V(y)$$

$$\iff W(x) = \phi(V(x)) \geq \phi(V(y)) = W(y)$$

■

Examples: The following functions represent the same order $\succeq$ on $X \subseteq \mathbb{R}^n$:

$$\text{(i)} \quad V(x) = \prod_{i=1}^{n} x_i^{\alpha_i},$$

$$\text{(ii)} \quad W(x) = \sum_{i=1}^{n} \alpha_i \cdot \ln x_i,$$

$$\text{(iii)} \quad Z(x) = A + \left(\sum_{i=1}^{n} \alpha_i\right)^{-1} \sqrt[5]{\prod_{i=1}^{n} x_i^{\alpha_i}}.$$

Utility functions that represent preferences are ordinal.

Other Axioms, $X \subseteq \mathbb{R}^n$

Axiom A4: *local nonsatiation*

For all $x \in X$ and all $\varepsilon > 0$ there exists $y \in X$ such that
$|x - y| < \varepsilon$ and $y \succ x$.

Axiom A4': *(weakly) monotone preferences*

For all $x, y \in X$ with $y \gg x$
$y \succ x$.

Axiom A4'': *strongly monotone preferences*

For all $x, y \in X$ with $y \geq x$ and $y \neq x$
$y \succ x$.

Axiom A5: *convex preferences*

For all $x \in X$,
$y, z \in B(x)$ and $\alpha \in [0,1]$
$\Rightarrow \quad \alpha \cdot y + (1-\alpha) \cdot z \succsim x.$

Axiom A5': *strictly convex preferences*

For all $x \in X$,
$y, z \in B(x), \quad y \neq z$ und $\alpha \in (0,1)$
$\Rightarrow \quad \alpha \cdot y + (1-\alpha) \cdot z \succ x.$

Quasi-concave utility functions:

A utility function $V : X \to \mathbb{R}$ is *quasi-concave*,
iff for all $\alpha \in \mathbb{R}$
the set $\{x \in X | \ V(x) \geq \alpha\}$ is *convex* or *empty*.

Denote by

$$I(x) := \{y \in X \mid \ y \sim x\}$$

the indifference set of $x \in X$.

Axiom A6: *homothethic preferences*

$$\begin{aligned} &\text{For all } x \in X, \\ &y \in I(x) \quad \text{and} \quad \alpha \geq 0 \\ &\Rightarrow \qquad \alpha \cdot y \in I(\alpha \cdot x). \end{aligned}$$