ODE vs PDE (Comprison of Ordinary and Partial differential equations)

1 Existence and uniqueness theorem for the ODE

2 Cauchy-Kovalevskaya theorem

3 Linear autonomous ODE and variables separation method

Consider a linear autonomous ODE:

$$\dot{x} = Ax, x \in \mathbb{R}^n, A : \mathbb{R}^n \to \mathbb{R}^n \tag{1}$$

a linear operator. The solutions form a linear space. If A has a real eigenbasis $\xi^1, ..., \xi^n$ with the eigenvalues $\lambda_1, ..., \lambda_n$, then the fundamental system of solutions (FSS, basis in the space of solutions) has the form

$$\varphi_j(t) = e^{\lambda_j t} \xi^j. \tag{2}$$

The Cauchy problem for equation (1) is:

$$\varphi(0) = x_0, \ x_0 \in \mathbb{R}^n. \tag{3}$$

The solution of the problem (1), (3) has the form:

$$x(t) = \Sigma_1^n c_j \varphi_j(t), \tag{4}$$

where

$$x_0 = \Sigma_1^n c_j \xi^j.$$

Consider now a PDE:

$$u_t = Au, \tag{5}$$

where A is a linear operator expressed through the drivatives in x. Let A have a real eigenbasis $X^1, ..., X^n, ...$ with the eigenvalues $\lambda_1, ..., \lambda_n, ...$. Then the FSS has the form:

$$u_j(t, x) = e^{\lambda_j t} X_j(x). \tag{6}$$

The general solution has the form:

$$u(t, x) = \Sigma_1^\infty c_j e^{\lambda_j t} X_j(x). \tag{7}$$

We ignore here the convergence problems. The Cauchy problem for equation (5) is:

$$u(0, x) = f(x). \tag{8}$$

The solution of the problem (5), (8) has the form (7) where

$$f(x) = \Sigma_1^\infty c_j X_j(x).$$

This a non-traditional justification of the variables separation method. If f belongs to a space spanned by the first n eigenfunctions of A, then the second Cauchy problem turns to be the same as the first one. Using the variables separation method we will solve heat, wave and Laplace equations on the circle.

4 Heat equation

$$u_t = \Delta u. \tag{9}$$

The Laplace operator Δ on the circle is just taking the second derivative in $x : \Delta u = u_{xx}$.

Lemma 1 *The Laplace operator on the circle has the eigenfunctions* $e^{ikx},\ k \in \mathbb{Z}$ *with the eigenvalues* $-k^2$.

Proof Solve Problem 1, List 1. ☐

Corollary 1 *General solution of the heat equation on the unit circle has the form:*

$$u(t,x) = \Sigma_{k\in\mathbb{Z}} c_k e^{ikx-tk^2}, c_k \in \mathbb{C}, \tag{10}$$

or

$$u(t,x) = a_0 + \Sigma_{k\in\mathbb{Z}} e^{-tk^2}(a_k \cos kx + b_k \sin kx),\ \ a_k,\ \ b_k \in \mathbb{R}.$$

5 Wave equation

$$u_{t^2} = u_{x^2},\ \ x \in S^1. \tag{11}$$

The same method provides a general solution:

$$u(t,x) = a_0 + b_0 t + \Sigma_{k\in\mathbb{Z}}(a_k e^{ikt+ikx} + b_k e^{-ikt+ikx}),\ \ a_k,\ \ b_k \in \mathbb{C}. \tag{12}$$

6 Laplace equation

$$u_{t^2} + u_{x^2} = 0,\ \ x \in S^1, t \in \mathbb{R}. \tag{13}$$

The same method provides a general solution:

$$u(t,x) = a_0 + b_0 t + \Sigma_{k\in\mathbb{Z}}(a_k e^{kt+ikx} + b_k e^{-kt+ikx}),\ \ a_k,\ \ b_k \in \mathbb{C}. \tag{14}$$

7 Cauchy problems

Cauchy problem for any of these equations is the union of the equation itself, and the initial data. Initial data for the heat equation has the form:

$$u|_{t=0} = \varphi(x), \; x \in S^1. \tag{15}$$

Initial data for the wave and Laplace equations has the form:

$$u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x), \; x \in S^1. \tag{16}$$

8 Convergence problems

The general solution of the heat equation on the circle is (10). Suppose that it converges at a point (t_0, x_0) with $t_0 < 0$. Then

$$|a_k| < Ce^{-k^2 t_0}$$

for some $C > 0$. Here a_k are Fourier coefficients of the initial data φ. This inequality fails even for generic analytic functions φ. For such functions the solution (10) diverges for any negative φ.

9 Enforced analitycity

Consider the Cauchy problem for the Laplace equation with the initial data $\varphi = \Sigma a_k e^{ikx}, \; \psi = 0$. The solution is

$$u(t, x) = a_0 + b_0 t + \Sigma_{k \in \mathbb{Z}} a_k e^{ikx} \operatorname{ch} kt.$$

Convergence at any point (t_0, x_0) with $t_0 \neq 0$ implies that

$$|a_k| < Ce^{-kt_0}.$$

Hence, φ is analytic. We conclude that the Cauchy problem above has a solution for analytic initial data only.

These examples show the drastic difference between the Cauchy problems for ODE and PDE, that is, between the finite and infinite dimensional phase space.