1 Nonlinear equations

1.1 Jet space.

Consider an equation

$$\Phi(x, u, Du) = 0, \ x \in \mathbb{R}^n, n > 1. \tag{1}$$

Definition 1 *Let u be a function of x. Then for any x the tuple $(x, u(x), Du(x))$ is a 1-jet of u at x.*

The set of 1-jets of u for all $x \in \Omega$ (domain of u), is the 1-graph of u.

Proposition 1 *The 1-graph of u is tangent to a hyperplane $du - pdx = 0$.*

Proof Indeed,

$$du = Dudx = pdx.$$

□

1.2 Contact structures.

Definition 2 *A contact structure on r^{2n+1} is a field of hyperplanes given by a one form α such that $\omega = d\alpha|_{\alpha=0}$ is nondegenerate: for any $\xi \in (\alpha = 0)$ there exists η such that $\omega(\xi, \eta) \neq 0$.*

$\alpha = du - pdx$ is an example of a contact structure.

Theorem 1 (Darboux) *Any contact structure is smooth equivalent to a standard one*

$$\alpha = du - pdx.$$

1.3 Skew-scalar product in $\mathbb{R}^{2n}$.

Definition 3 *A skew scalar product is a bi-linear skew symmetric non=degenerate form on a linear space.*

Definition 4 *Non-degeneracy $\Rightarrow$ a skew scalar product exists on $\mathbb{R}^{2n}$ only.*

Contact structure: $\exists$ on $\mathbb{R}^{2n+1}$ only. The contact structure defines a field of hyperplanes: $\alpha = 0$. A hyperplane of this field attached at a point a is defined by Π_a.

Definition 5 *A plane of dimension n skew orthogonal to itself is called Lagrangian.*

Proposition 2 *There exists no plane of dimension $n + 1$ in a $2n$-symplectic space, skew orthogonal to itself.*

1.4 Cauchy problem for non-linear first order PDE: an overlook.

Take an equation (1). It defines a surface $E = E^{2n}$ in the jet space. Suppose that it is smooth.

Definition 6 *A characteristic point $a \in E$ is such that $T_aE = \Pi_a$.*

Suppose that E has no characteristic points. At any point $a \in E$ consider the contact plane Π_a and a form $\omega = d\alpha = dp \wedge dx$ restricted to Π_a. Note that (p, x) are coordinates on Π_a.

Let $P_a = \Pi_a \cap T_aE$ be *the characteristic* $(2n-1)$-plane. Let L be a line field $a \mapsto l_a \in P_a$ such that $l_a \perp P_a$ in sense of the skew-scalar product in Π_a. (My tex has no "skew orthogonal" symbol, so I use "orthogonal", YuI). Integral curves of this field are called *characteristics of the surface E.*

Theorem 2 *Any 1-graph of a solution to (1) is saturated by the characteristics of the corresponding surface E.*

The initial data is as usual:

$$u|_\gamma = \varphi. \tag{2}$$

γ and φ are the same as before.

Solution to the Cauchy problem consists of three steps:

1. Lift the graph of the initial data to the surface E of the equation.
2. Saturate it by the characteristics of the surface E.
3. Project the 1-graph of the solution thus obtained to the (x, u) space, and thus obtain the graph of the solution.

Remark 1 *This construction proves uniqueness: if the solution exists, it is unique. Existence requires more ideas.*

2 Proof of Theorem 2

Proof The function is a solution to equation (1) iff its one-graph Γ belongs to E. Hence, the tangent planes to Γ are Lagnangian, because $\Gamma \subset \Pi_a \ \forall a \in \Gamma, \ \alpha|_\Gamma \equiv 0$. This implies that for any two vectors $\xi, \eta \in T_a\Gamma, \ \omega(\xi, \eta) = 0$. Hence, ξ, η are skew orthogonal. But l_a is skew orthogonal to $P_a \supset T_a\Gamma$. Hence, the space spanned by $T_a\Gamma$ and l_a is skew orthogonal to itself. If $l_a \notin T_a\Gamma$, then the demension of this space is $n+1$, a contradiction. The Invariant Manifold Lemma of Lecture 3 now implies the theorem. $\square$

3 Lift of the initial data

Any point $x \in \gamma$ is lifted to E in the following way to (x, u, Du):

1. x is given
2. $u = \varphi(x)$
3. $du|_{T_x\gamma} = d\varphi$
4. $(x, u, Du) \in E$.

This may be done sometimes, because $\delta\varphi$ provides $n - 1$ partial derivatives of u at x, and the n-th one is found from the equation. We suppose that this may be done.

4 Solution to the Cauchy problem

We saturate the lift of the initial data by characteristics, and obtain the 1-graph of the solution. The latter statement should be justified.

Theorem 3 *The saturation of the lifted initial data by the characteristics near ant non-characteristics point is tangent to the characteristic, hence, contact, planes.*

This is non-sufficient for an n-surface to be a 1-graph. It is also needed that the projection of this surface to the x-plane along the (u, p)-plane is bijective.

A point $x \in \gamma$ is characteristic for the problem (1), (2) provided that the projection of the characteristic line at the point $(x, \varphi(x), p)$ of the graph of the lifted initial data along the (u, p) plane is tangent to the initial surface at the point x. .

Theorem 4 *In some neighborhood of any non-characteristic point, the problem* (1), (2) *has a unique solution.*

Proof The proof is given modulo Theorem 3. Let $\Gamma^{n-1} \subset E$ be the lifted initial data, $a = (x, \varphi(x), p) \in \Gamma^{n-1}$, L be the saturation of Γ_{n-1} by characteristics. Then the tangent plane T_aL is projected to $T_x\gamma \oplus l_x = T_x\mathbb{R}^n$, where l_x is the projection of l_a to $\mathbb{R}^n$. By the inverse function theorem, L has the form: $u = f(x), p = g(x)$. By Theorem 3, L is tangent to the contact planes. Hence $\alpha|_L = 0$. That is, on L

$$du = pdx.$$

This implies that $p = Du$, and L is a 1-graph. □

Theorem 3 will be proved in the next lecture.

5 Explicite form of the characteristic equation

Characteristic vector field for equation (1) has the form:

$$\begin{aligned}\dot{x} =&\Phi_p\\ \dot{p} =& -(\Phi_x + \Phi_u p)\\ \dot{u} =&p\Phi_p.\end{aligned}$$

The proof may be found in the students lecture notes, or in the “Geometric methods...” by Arnold.