Lecture 2-3

Dr. Alexander Karpov

Decisions under uncertainty

- **von Neumann & Morgenstern**
  - Expected Utility (EU):
  - choice of lotteries
- **Savage**
  - Subjective Expected Utility (SEU)
  - choice of actions, i.e. state-contingent outcomes

Decisions under uncertainty

Objects of the decision are

– *probability distributions over outcomes (lotteries)*

– *state-contingent outcomes*.

• Preferences order a set of lotteries or a set of state-contingent outcomes.

It is usual to distinguish

• *decisions under risk:*

– probabilities of outcomes are part of the information of the decision maker, i.e. objects of the decision are *lotteries,*

• *decisions under uncertainty:*

– probabilities of outcomes are *not part of the information* of the decision maker, i.e. objects of the decision are *state-contingent outcomes*.

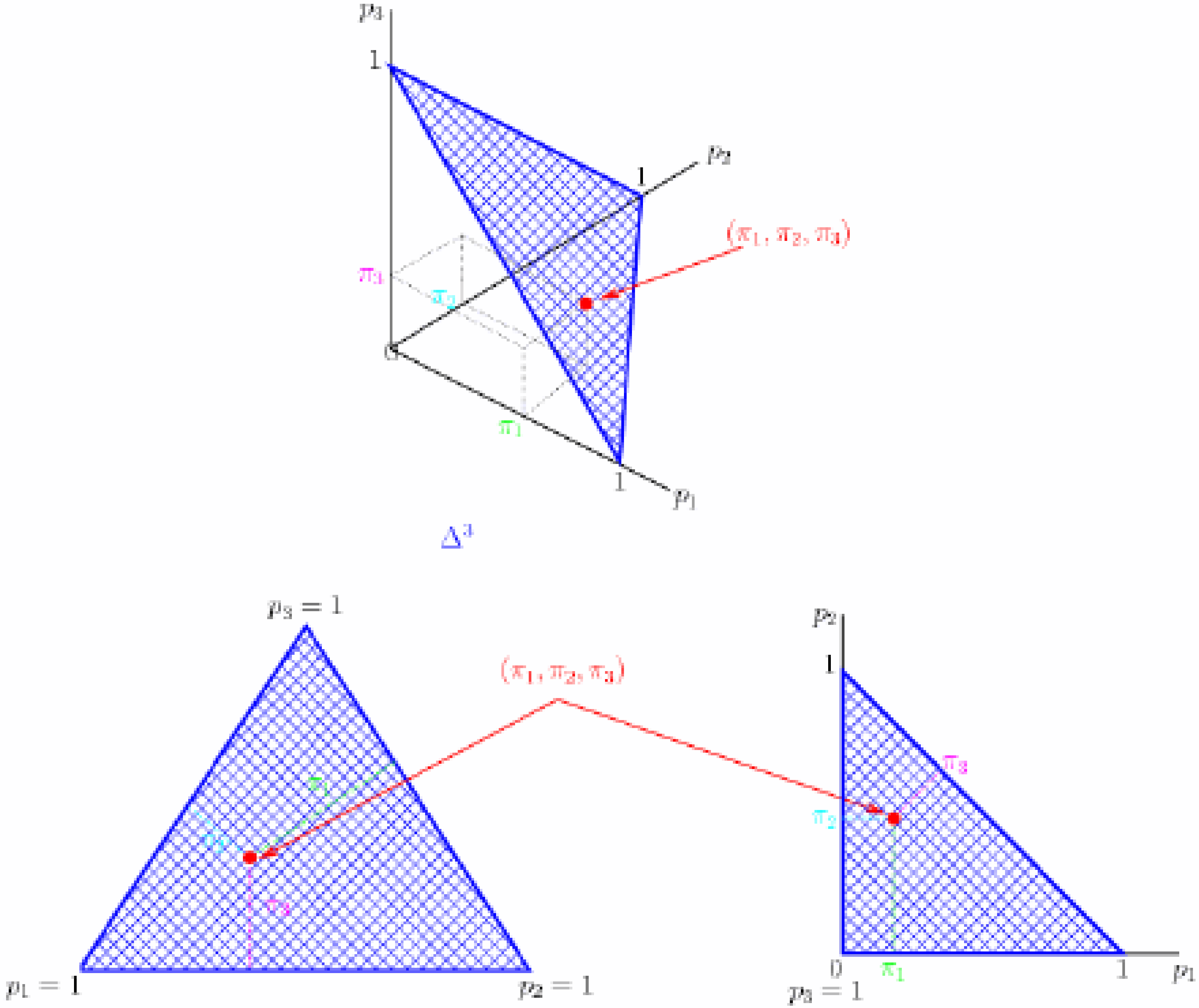
Choice over lotteries: von Neumann-Morgenstern approach

Consider a finite set of outcomes:

$$X := \{x_1, ..., x_n\}.$$

The set of probabilities over the outcomes in X is

$$\Delta^n := \{(p_1, ..., p_n) \in \mathbb{R}^n_+ | \sum_{i=1}^{n} p_i = 1\} \subset \mathbb{R}^n.$$



p_3
1
p_2
1
(π_1, π_2, π_3)
π_3
π_2
0
π_1
1
p_1
Δ^3
$p_3 = 1$
(π_1, π_2, π_3)
π_1
π_2
π_3
$p_1 = 1$
$p_2 = 1$
p_2
1
π_3
π_2
0
π_1
1
p_1
$p_3 = 1$

Expected Utility

One seeks axioms (assumptions) which guarantee the existence of an expected utility representation:

$$V(p) = \sum_{i=1}^{n} p_i \cdot u(x_i)$$

It is important to distinguish the utility function U(), defined on lotteries, and the utility functionn u() defined on sure amounts of money. For this reason we call U() the von-Neumann-Morgenstern expected utility function and u() the Bernoulli utility function.

Properties of EU function

$$V(p) = \sum_{i=1}^{n} p_i \cdot u(x_i)$$

von Neumann-Morgenstern utilities are unique up to a linear affine transformation:

for $b > 0$, $w(x) = a + b \cdot u(x)$.

w and u represent the same preferences.

- One can normalise two values of the von Neumann-Morgenstern utility function: for $u(x^*) > u(x_*)$, define

$$v(x) := \frac{u(x) - u(x_*)}{u(x^*) - u(x_*)},$$

then

$$v(x^*) = 1 \text{ and } v(x_*) = 0.$$

Convex combination of lotteries

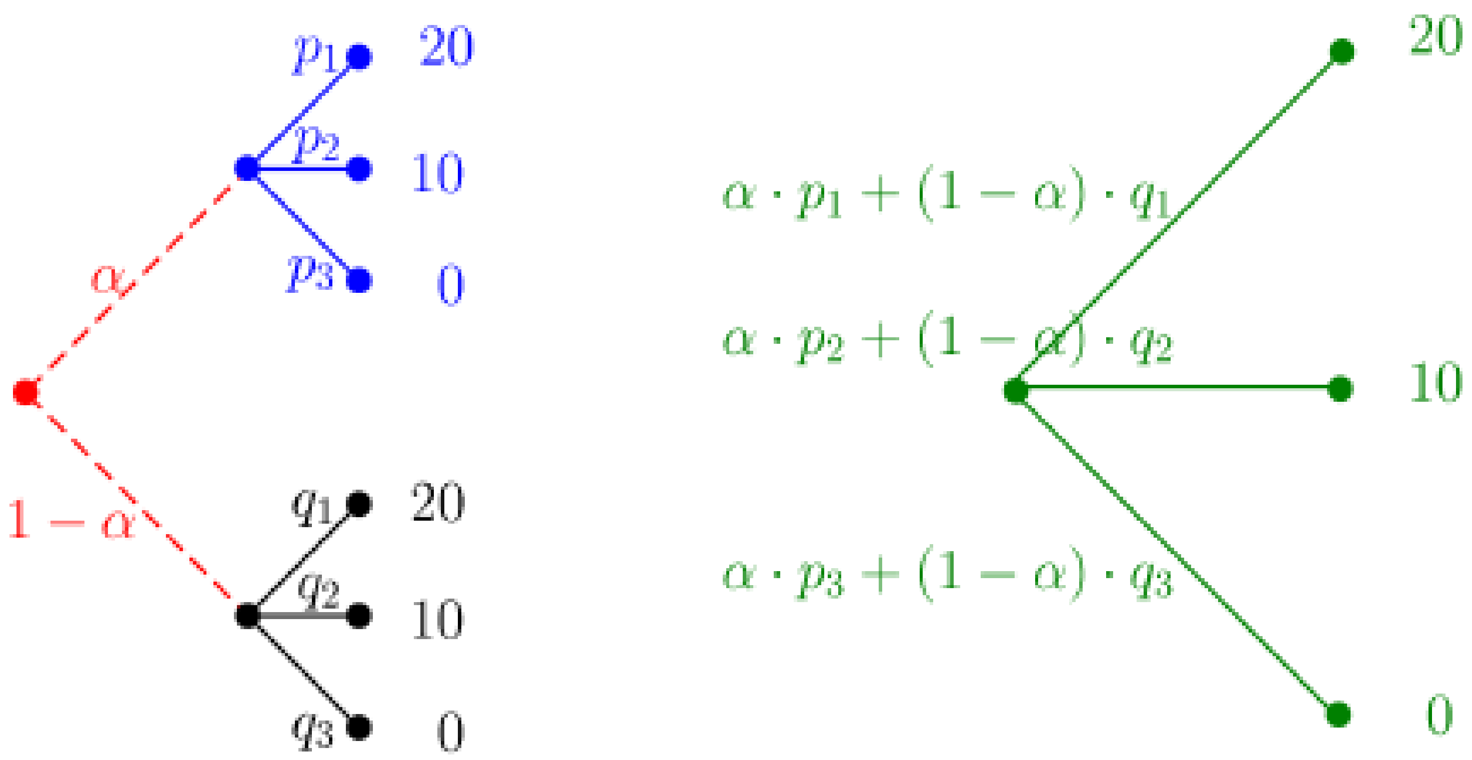
A convex combination of two lotteries $p,q \in \Delta^n$ is defined as

$$\alpha \cdot p + (1-\alpha) \cdot q \; = \; \alpha \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} + (1-\alpha) \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

$$= \; \begin{pmatrix} \alpha \cdot p_1 + (1-\alpha) \cdot q_1 \\ \vdots \\ \alpha \cdot p_n + (1-\alpha) \cdot q_n \end{pmatrix}$$

for any $\alpha \in [0,1]$.

- One may interpret convex combinations of lotteries as compound lotteries.



Axioms of preference order

Consider a set X and a preference order ≥ on X.

Axiom A1: *Completeness*

For all $x, y \in X$,

either $x \succeq y$ or $y \succeq x$.

Axiom A2: *Transitivity*

For all $x, y, z \in X$,

$x \succeq y$ and $y \succeq z$ imply $x \succeq z$.

Continuity

Axiom A3b: *Continuity*

For all $x \in X$,

$B(x)$ and $W(x)$ are *closed sets* in X.

Remark: *Continuity* in Axiom **3b** is equivalent to the familiar notion of continuity:

$$\begin{array}{lll} \text{(i)} & x^n \in B(x) \ \text{ for all } n & \text{and} \\ \text{(ii)} & x^n \to x^0 & \\ & \text{implies} \quad x^0 \in B(x). & \end{array}$$

Independence axiom

Axiom A8: *Independence*

For all $p, q, r \in \Delta^n$ and all $\alpha \in [0,1]$,

$$\alpha \cdot p + (1-\alpha) \cdot r \succeq \alpha \cdot q + (1-\alpha) \cdot r \quad \Leftrightarrow \quad p \succeq q.$$

Theorem

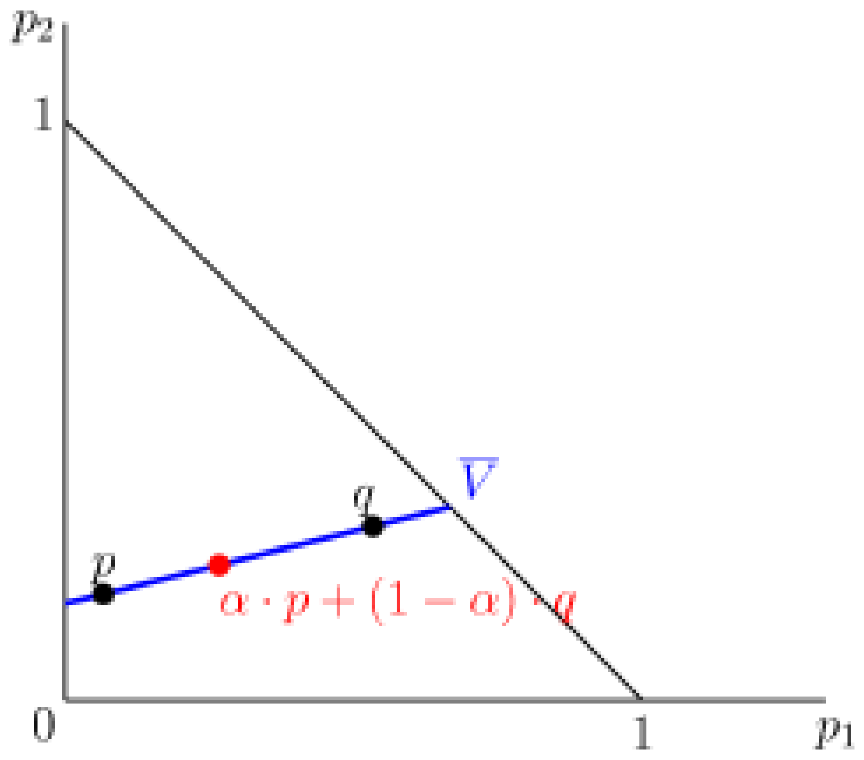
Proposition: The following statements are equivalent:

(i) The preference order $\succeq$ on $\Delta^n \subset \mathbb{R}^n$ satisfies Axioms **A1, A2, A3b**, and **A8.**

(ii) There exists a function $u : X \to \mathbb{R}$ such that for all $p, q \in \Delta^n$,

$$p \succeq q \quad \Leftrightarrow \quad \sum_{i=1}^{n} p_i \cdot u(x_i) \geq \sum_{i=1}^{n} q_i \cdot u(x_i).$$

Proof



(i) Linear indifference curves:

Suppose

$$V(p) = V(q) = \overline{V}.$$

Then, by **A8**,

$$\begin{aligned} V(\alpha p + (1-\alpha)q) &= V(\alpha p + (1-\alpha)q) \\ &= V(q) = \overline{V}. \end{aligned}$$

(ii) Parallel indifference curves:

Suppose

$$V(p) = V(q) = \overline{V} > V(r).$$

For any $\widetilde{V}$ such that

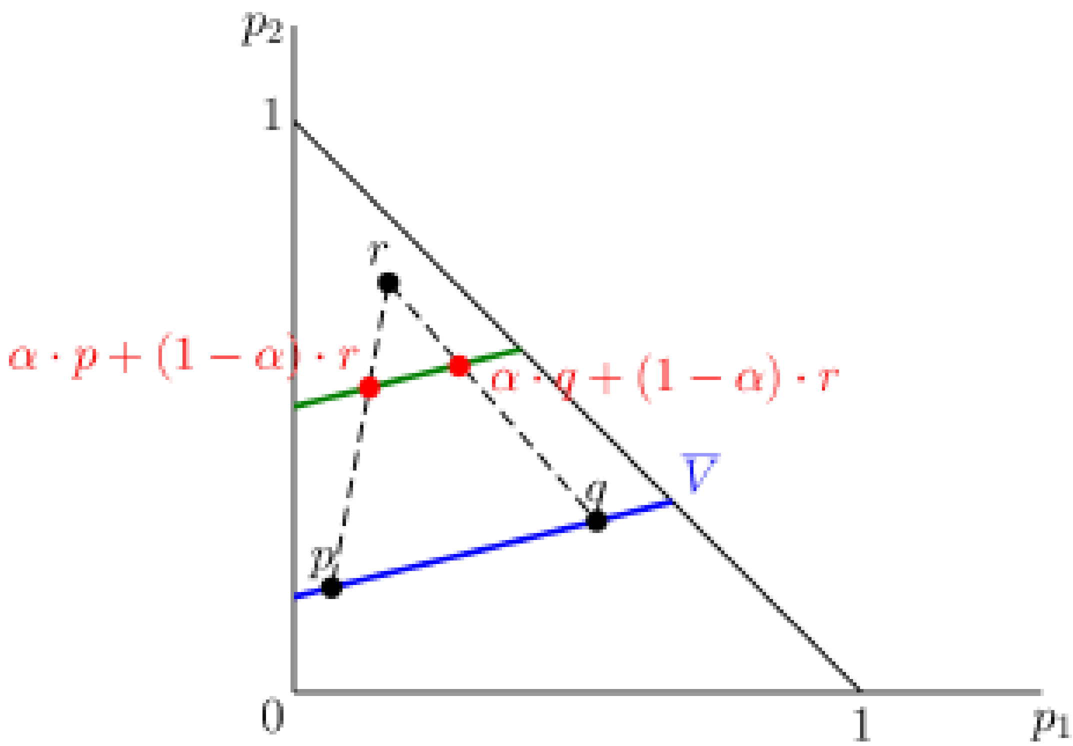
$$\overline{V} = V(p) > \widetilde{V} > V(r)$$

by continuity (**A3b**) there exists $\alpha \in (0, 1)$ such that

$$V(\alpha p + (1 - \alpha)r) = \widetilde{V}.$$

By **A8**,

$$V(\alpha q + (1 - \alpha)r) = V(\alpha p + (1 - \alpha)r) = \widetilde{V}.$$



(iii) Expected utility representation:

A family of parallel linear surface in $\mathbb{R}^n$ is uniquely defined by a normal vector $u = (u_1, ..., u_n) \in \mathbb{R}^n$ such that

$$p \cdot u = \sum_{i=1}^{n} p_i u_i = k.$$

For any $p \in \Delta^n$ define

$$\widehat{V}(p) \quad : \quad = \sum_{i=1}^{n} p_i u_i$$
$$= \sum_{i=1}^{n} p_i u(x_i).$$

$\widehat{V}$ represents the same preferences over Δ^n as V. ■

Uniqueness

A von Neumann-Morgenstern utility function

$$u : X \to \mathbb{R}$$

is unique up to a *positive linear affine transformation*,

$$v(x) := a + b \cdot u(x), \quad b > 0.$$

- For *positive linear affine transformations* of the utility function u the marginal rate of substitution between any two outcomes remains unchanged:

Problem 1

Consider a set of simple lotteries Δ and a preference order $\succeq$ on Δ. $\succeq$ are complete, transitive, and satisfy independence axiom. Prove that for all lotteries $L_1, L_1 \in \Delta, L_1 \succ L_2$ and any $\alpha, \beta \in (0,1)$ we have

$$\beta > \alpha \quad iff \quad \beta L_1 + (1-\beta) L_2 \succ \alpha L_1 + (1-\alpha) L_2$$

Problem 2

xercise 6.B.1: Show that if the preferences $\succsim$ over $\mathscr{L}$ satisfy the independenc iom, then for all $\alpha \in (0,1)$ and $L, L', L'' \in \mathscr{L}$ we have

$$L \succ L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

id

$$L \sim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

iow also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

1.2 Risk

- Consider a lottery

$$L = \langle x_1, \ldots, x_n; p_1 \ldots, p_n \rangle \, .$$

- *Examples:*

(i) Urns with balls of differing colours on which one can bet.

(ii) Coin tossing.

Risk attitudes

- The ***certainty equivalent*** of a lottery L is the amount of money Q_L which a decision maker would consider equivalent to the lottery,

$$u(Q_L) = \sum_{i=1}^{n} p_i u(x_i).$$

- The ***risk premium*** is the amount of money P_L which a decision maker would be willing to pay if the expected value of the lottery would be paid out rather than having to play the lottery,

$$u(\sum_{i=1}^{n} p_i x_i - P_L) = \sum_{i=1}^{n} p_i u(x_i).$$

Risk attitudes

- A*risk-averse* decision maker will
  - pay a *positive risk premium* P_L an
  - has a *certainty equivalent* Q_L which *exceeds the expected value of the lottery* L,

$$P_L > 0 \Leftrightarrow Q_L > \sum_{i=1}^{n} p_i x_i.$$

- A*risk-loving* decision maker will
  - pay a *negative risk premium* P_L and
  - has a *certainty equivalent* Q_L which is *lower than the expected value of the lottery* L,

$$P_L = 0 \Leftrightarrow Q_L = \sum_{i=1}^{n} p_i x_i.$$

Risk attitudes

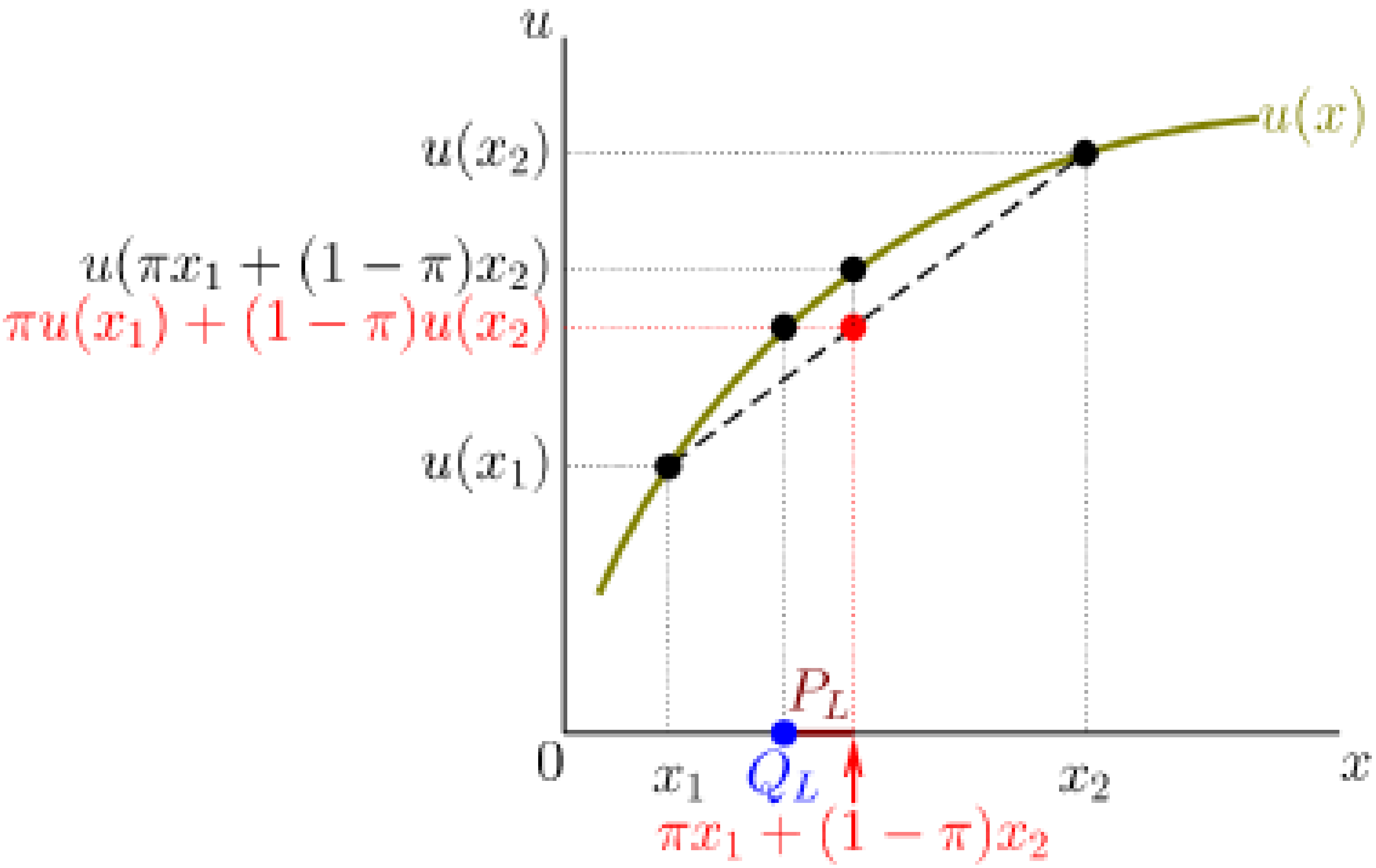
- A *risk-neutral* decision maker will
  - pay *no risk premium* P_L and
  - has a *certainty equivalent* Q_L which *equals the expected value of the lottery* L,

$$P_L = 0 \Leftrightarrow Q_L = \sum_{i=1}^{n} p_i x_i .$$

- *Risk attitudes* can be characterised by the *curvature* of the von Neumann-Morgenstern utility function u(•):

risk attitude	**curvature of $u(\cdot)$**	**risk premium**
risk averse	concave	positive
risk neutral	linear	null
risk loving	convex	negative

Figure



Measures of risk aversion

- If the von Neumann-Morgenstern utility function u(x) is twice continuously differentiable, then one can measure the degree of risk aversion by the second derivative:

- *absolute risk aversion:* $R_a(x) := -\frac{u''(x)}{u'(x)}$.
- *relative risk aversion:* $R_r(x) := -x \cdot \frac{u''(x)}{u'(x)}$.

Special cases

- *constant absolute risk aversion*:

$$R_a(x) = b > 0: \quad u(x) = -e^{-b \cdot x}.$$

- *constant relative risk aversion*:

$$R_r(x) = b < 1: \quad u(x) = (1-b) \cdot x^{1-b},$$

$$R_r(x) = b = 1: \quad u(x) = \ln x.$$

1.2.2 Stochastic dominance and expected utility

- A random variable X, e.g., a lottery L, can be represented by a (cumulative) distribution function,

$$F(x) := \Pr(\widetilde{X} \leq x).$$

- A distribution function $F : \mathbb{R} \to [0, 1]$ is an increasing function of x.

Orderings

Distribution functions can be partially ordered.

- One distinguishes
  - first order stochastic dominance, and
  - second order stochastic dominance.

First order stochastic dominance

Definition: A probability distribution *F dominates another* probability distribution *G* according to the *first order stochastic dominance*, if for all *x*

$$F(x) \leq G(x).$$

Theorem: A probability distribution F *dominates* G *according to first order stochastic dominance*, if and only if

$$\int u(x)\,dF \geq \int u(x)\,dG$$

for all *strictly increasing* utility functions *u*.

Example

Example: *First order stochastic dominance*

$$L_1(x) = \begin{cases} 1 & \text{for } 20 \leq x \\ \frac{1}{2} & \text{for } 10 \leq x < 20 \\ 0 & \text{for } x < 10 \end{cases},$$

$$L_2(x) = \begin{cases} 1 & \text{for } 20 \leq x \\ \frac{3}{4} & \text{for } 10 \leq x < 20 \\ 0 & \text{for } x < 10 \end{cases}.$$

$$L_1 \succ L_2$$

Example

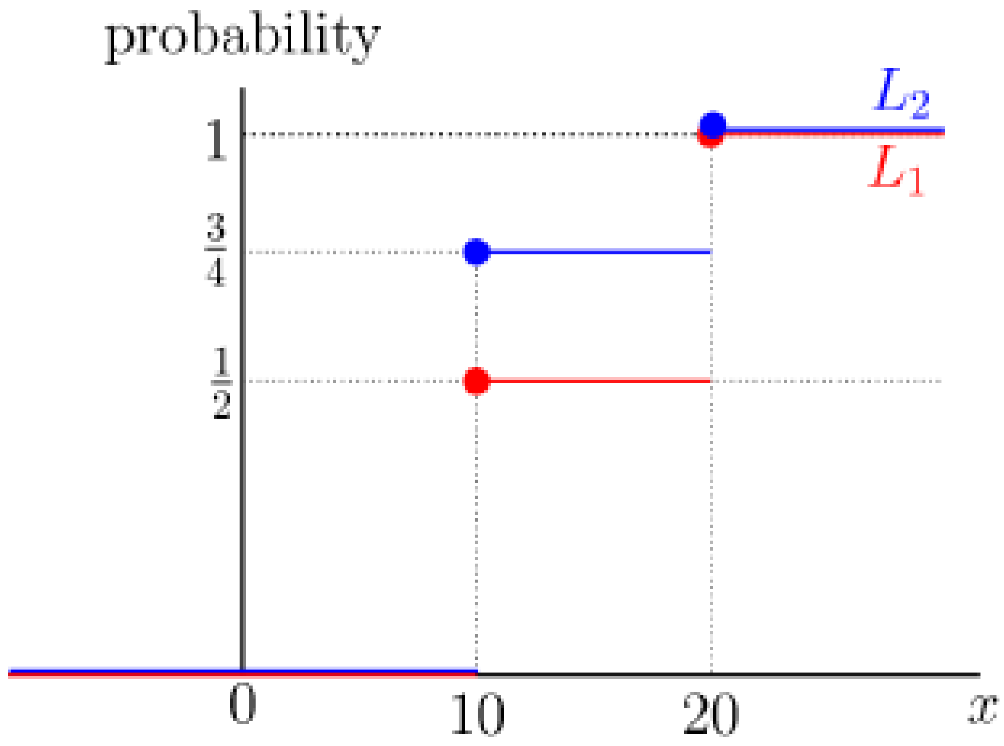


Figure 3.

$L_1 \succ L_2$

Second order stochastic dominance

Definition: A probability distribution F *dominates* another probability distribution G according to *second order stochastic dominance*, if for all x

$$\int_{-\infty}^{x} F(t)\,dt \leq \int_{-\infty}^{x} G(t)\,dt.$$

Theorem: A probability distribution F *dominates* G according to *second order stochastic dominance*, if and only if

$$\int u(x)\,dF \geq \int u(x)\,dG$$

for all *strictly increasing* and *concave* utility functions *u*.

Example

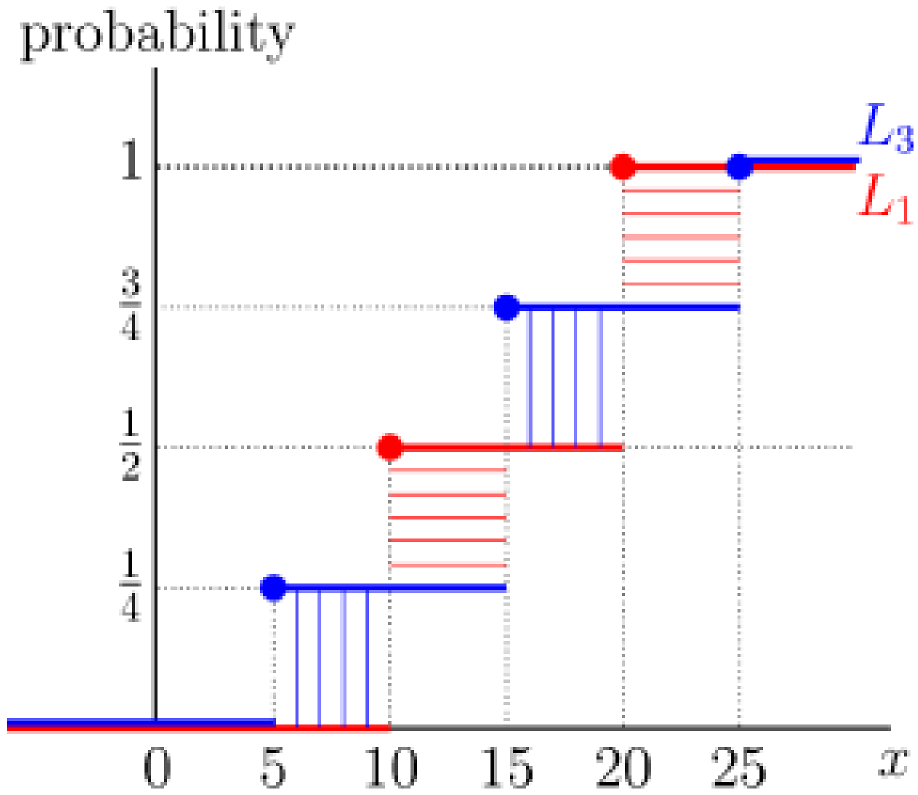
Example: *Second order stochastic dominance*

$$L_1(x) = \begin{cases} 1 & \text{for } 20 \le x \\ \frac{1}{2} & \text{for } 10 \le x < 20 \\ 0 & \text{for } x < 10 \end{cases},$$

$$L_3(x) = \begin{cases} 1 & \text{for } 25 \le x \\ \frac{3}{4} & \text{for } 15 \le x < 25 \\ \frac{1}{4} & \text{for } 5 \le x < 15 \\ 0 & \text{for } x < 5 \end{cases}.$$

$$L_1 \succ L_3$$

Example



$L_1 \succ L_3$

Special case: *mean-preserving spread*

$$\int_{-\infty}^{x} F(t)\,dt \leq \int_{-\infty}^{x} G(t)\,dt$$

and

$$\int_{-\infty}^{\infty} t dF = \int_{-\infty}^{\infty} t dG.$$

Risk aversion

Definition 1

An agent is **risk-averse** if, at any wealth level w, he or she dislikes every lottery with an expected payoff of zero: $\forall w$, $\forall z$ with Ez=0,

$$\mathrm{E}u(w+z) \leqslant u(w).$$

For any lottery z and for any initial wealth w,

$$\mathrm{E}u(w+z) \leqslant u(w+Ez).$$

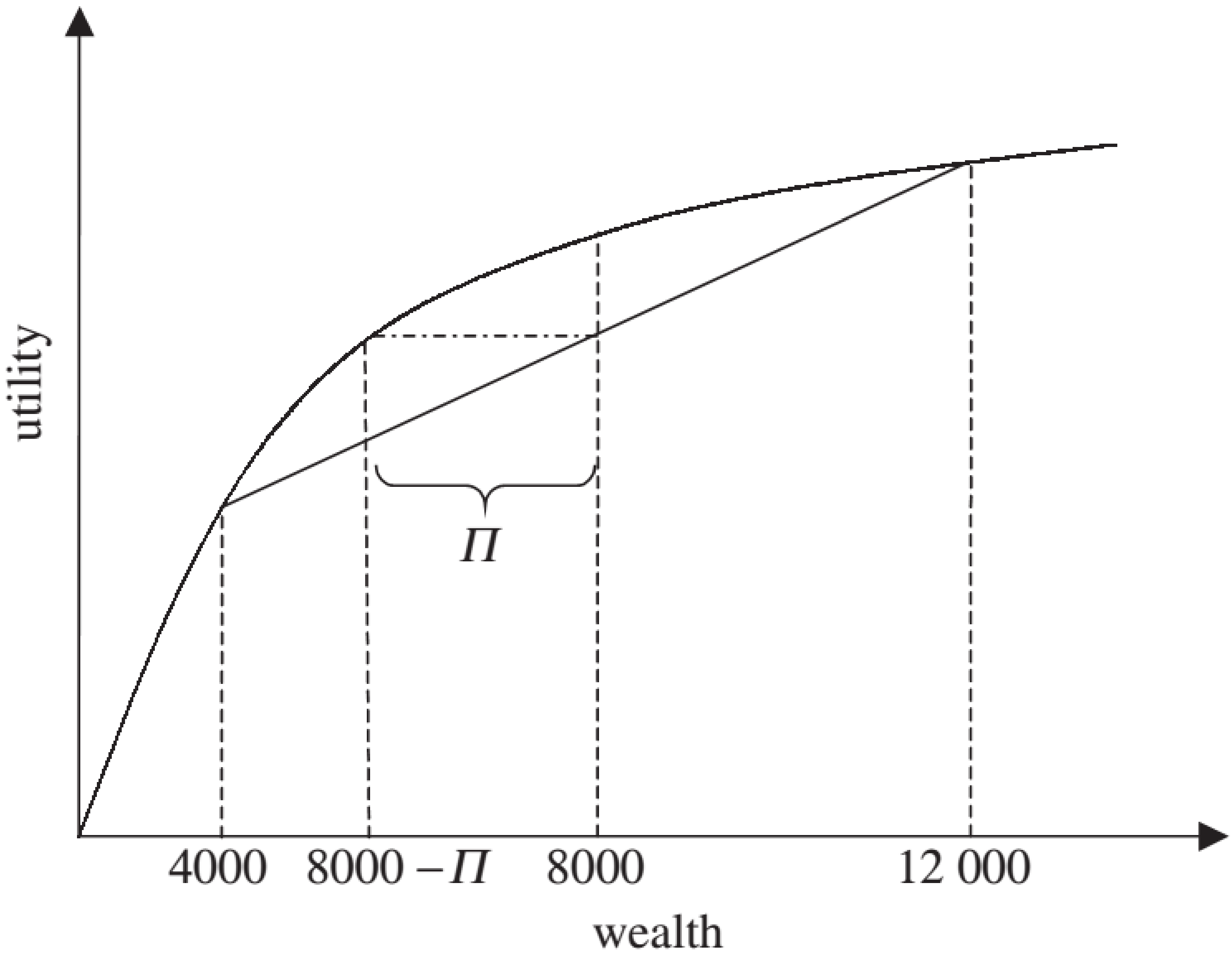
Proposition 1

A decision maker with utility function u is risk-averse, if and only if u is concave.

Behavioral consequences

- Risk averse agent will always purchase full insurance at an actuarially fair price.
- Risk averse agent will not play a lottery game (even fair).

Risk premium Π.



Degree of absolute risk aversion

The degree of absolute risk aversion of the agent evaluated at *w*

$$A(w) = \frac{-u''(w)}{u'(w)}.$$

Under risk aversion, function A is positive. It would be zero or negative respectively for a risk-neutral or risk-loving agent.

Absolute risk aversion measures the rate at which marginal utility decreases when wealth is increased by one euro.

The Arrow–Pratt approximation of risk premium

The cost of risk, as measured by the risk premium, is approximately proportional to the variance of its payoffs (for zero-mean lottery). Thus, the variance might appear to be a good measure of the degree of riskiness of a lottery.

$$\Pi \simeq \frac{1}{2}\sigma^2 A(w),$$

Degree of absolute risk aversion

Let us assume that $z = ke$, with Ee=0.

We obtain that

$$\Pi(k) \simeq \frac{1}{2} k^2 \sigma_{\tilde{\varepsilon}}^2 A(w),$$

Π'(0) =0. At the margin, accepting a small zero-mean risk has no effect on the welfare of risk-averse agents.

Comparative risk aversion

Definition 2.

Suppose that agents u and v have the same wealth w, which is arbitrary. An agent v is more risk-averse than another agent u with the same initial wealth if any risk that is undesirable for agent u is also undesirable for agent v. In other words, the risk premium of any risk is larger for agent v than for agent u.

Proposition 2

The following three conditions are equivalent.

(a) Agent v is more risk-averse than agent u, i.e. the risk premium of any risk is larger for agent v than for agent u.

(b) For all w, $A_v(w) \geqslant A_u(w)$.

(c) Function v is a concave transformation of function u: $\exists \varphi(\cdot)$ with $\varphi' > 0$ and $\varphi'' \leqslant 0$ such that $v(w)=\varphi(u(w))$ for all w.

Example

Comparative risk aversion

$$v(w)=\ln(w),$$

$$u(w)=w^{0.5}.$$

Decreasing Absolute Risk Aversion and Prudence

A lottery to gain or lose 100 with equal probability is potentially life-threatening for an agent with initial wealth w=101, whereas it is essentially trivial for an agent with wealth w=1 000 000.

Definition 3.

The risk premium $\Pi=\pi(w)$ as a function of initial wealth w can be evaluated by solving

$$Eu(w+z)=u(w-\pi(w)).$$

Proposition

The risk premium associated to any risk *z* is decreasing in wealth if and only if absolute risk aversion is decreasing; or equivalently if and only if prudence is uniformly larger than absolute risk aversion.

Prudence $P(w) = -u'''(w)/u''(w)$

Because $A'(w) = A(w)[A(w) - P(w)]$,

$P(w) \geqslant A(w)$ is equivalent to condition to the condition A'(w)$\leqslant$0.

Relative Risk Aversion

The index of absolute risk aversion is not unit free, as it is measured per euro (per dollar, or per yen).

Definition 4. Define the index of relative risk aversion R as the rate at which marginal utility decreases when wealth is increased by one percent.

$$R(w) = -\frac{\mathrm{d}u'(w)/u'(w)}{\mathrm{d}w/w} = \frac{-wu''(w)}{u'(w)} = wA(w).$$

In terms of standard economic theory, this measure is simply the wealth-elasticity of marginal utility.

$$\hat{\Pi}(\tilde{z}) = \frac{\Pi(w\tilde{z})}{w} \simeq \frac{\frac{1}{2}w^2\sigma^2 A(w)}{w} = \frac{1}{2}\sigma^2 R(w).$$

Properties of successive derivatives

- Eeckhoudt, Louis, and Harris Schlesinger. 2006. "Putting Risk in Its Proper Place." *American Economic Review*, 96(1): 280-289.
- Crainich, David, Louis Eeckhoudt, and Alain Trannoy. 2013. "Even (Mixed) Risk Lovers Are Prudent." *American Economic Review*, 103(4): 1529-35.

N^{th}-degree stochastic dominance

The signs of the first n derivatives of utility coincide with a preference for n^{th}-degree stochastic dominance.

Definition 5. Define $F^{(1)}(x)=F(x)$, and then define $F^{(i)}(x)=\int_a^x F^{(i-1)}(t)dt$ for all $i\geq 2$.

Definition 6. The distribution G is an Nth-degree increase in risk over F if $F^{(N)}(x)\leq G^{(N)}(x)$, for all $a \leq x \leq b$ and $F^{(N)}(a)\leq G^{(N)}(b)$, for $i= 2; \ldots; N-1$.

Interpretation of successive derivatives

- $u'' < 0$ risk aversion
- $u''' > 0$ prudence
- $u^{iv} < 0$ temperance
- An individual dislikes two things: a certain reduction in wealth ($-k$) and adding a zero-mean independent noise random variable ($\tilde{\varepsilon}$) to the distribution of wealth.

Prudence

An individual is said to be prudent if the lottery B3 =[-k; $\tilde{\varepsilon}$] is preferred to the lottery

A3= [-k+$\tilde{\varepsilon}$, 0], where all outcomes of the lotteries have equal probability, for all initial wealth levels x and for all k and all $\tilde{\varepsilon}$.

For prudent individual it is better to attach additional risk to the better outcome ***0****, than to the outcome* ***-k****.*

Prudence

- This logic helps to explain why someone opts for a higher savings when second-period income is risky in a two-period model. The resulting higher wealth in the second period helps one to cope with the additional risk.
- Prudence as equivalent to a precautionary demand for savings.
- Even (Mixed) Risk Lovers Are Prudent.

Temperance

An individual is said to be temperate if the lottery B3 =[$\widetilde{\varepsilon_1}$; $\widetilde{\varepsilon_2}$] is preferred to the lottery A3= [$\widetilde{\varepsilon_1}$+$\widetilde{\varepsilon_2}$, 0], where all outcomes of the lotteries have equal probability, for all initial wealth levels x and for all k and all $\widetilde{\varepsilon_1}$, $\widetilde{\varepsilon_2}$.

For temperateindividual it is better to attach additional risk to the better outcome ***0****, than to the outcome* $\widetilde{\varepsilon_1}$.

Some Classical Utility Functions

$$u(w) = aw - \tfrac{1}{2}w^2, \quad \text{for } w \leqslant a.$$

$$Eu(\tilde{w}) = aE\tilde{w} - \tfrac{1}{2}E\tilde{w}^2.$$

In this case, the EU theory simplifies to a mean–variance approach to decision making under uncertainty.

$$A(w) = \frac{1}{a - w} \Rightarrow A'(w) = \frac{1}{(a - w)^2} > 0.$$

The quadratic utility functions exhibit increasing absolute risk aversion. For this reason, quadratic utility functions are not as in fashion anymore.

Constant-absolute risk-aversion (CARA) utility function

$$u(w) = -\frac{\exp(-aw)}{a},$$

where a is some positive scalar.

$$A(w)=a \text{ for all } w.$$

Risk whose size is invariant to changes in wealth.

Constant-relative-risk-aversion (CRRA) utility function

$$u(w) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} & \text{for } \gamma \geqslant 0,\ \gamma \neq 1, \\ \ln(w) & \text{for } \gamma = 1. \end{cases}$$

$A(w) = \gamma / w$ and $R(w) = \gamma$ for all w.

This class of utility functions eliminates any income effects when making decisions about risks whose size is proportional to one's level of wealth. The assumption that relative risk aversion is constant enormously simplifies many of the problems often encountered in macroeconomics and finance.

An Application: The Cost of Macroeconomic Risks

Suppose that we attach an equal probability that any of the annual growth rate observed during the period 1963 to 1992 occurs next year.

RRA	Certainty equivalent growth rate (%)	Social cost of macroeconomic risk (%)
$\gamma = 0$	1.86	—
$\gamma = 0.5$	1.85	0.01
$\gamma = 1$	1.83	0.03
$\gamma = 4$	1.74	0.12
$\gamma = 10$	1.56	0.30
$\gamma = 40$	0.50	1.36

$$E\,\frac{(100+g)^{1-\gamma}}{1-\gamma}$$