1 Changes of variables

1.1 Existence of solutions of the 1 order PDE's.

Lemma 1 *The field of planes $\alpha = 0$ on $E (\alpha \neq 0$ on $E)$ is invariant under the phase flow of a vector field v such that $\alpha(v) = 0$, provided that for any ξ such that $\alpha(\xi) = 0$, we have:*

$$d\alpha(v, \xi) = 0.$$

Proof Let us rectify $v : v \equiv e_1$. Let $\alpha = \Sigma a_j dx_j$. As $\alpha(v) \equiv 0$, we have: $a_1 \equiv 0$. As $\alpha \neq 0$ on E, $\forall x \exists j : a_j(x) \neq 0$. Let $j = 2$. Take any other j, and let $\xi = (0, -a_j, 0, \ldots, a_2, 0, \ldots)$. Let $\partial a_j / \partial x = \dot{a}_j$. Then $d\alpha(v, \xi) = -\dot{a}_2 a_j + a_2 \dot{a}_j$. Hence, $\frac{a_j}{a_2} = \text{const}$ along v. The form α may change along the orbits of v, but the field $\alpha = 0$ may not. □

Existence is proved now in a few words.

1.2 Changes of variables in the operator.

Consider a change of variables $\xi = \xi(x)$.

The chain rule implies $D_x u^t = j^* D_\xi u^t$.

Consider a linear differential operator of the first order:

$$L_x u = (\nabla_x, A(x) \nabla_x) u.$$

After a change of variables, we have:

$$L_\xi u = (J^* \nabla_\xi, A(x) J^* \nabla_\xi) = (\nabla_\xi, J A(x) J^* \nabla_\xi) + \text{ lower terms(if} J \neq \text{const}).$$

The higher order terms are changed in the following way:

$$A(x) \rightsquigarrow \tilde{A} = J(x) A(x) J^*(x).$$

Consider analogous formulas from algebra.

1.3 Change of variables in the quadratic form.

Let

$$F_A(y) = (y, Ay), \; y = Cz.$$

Then

$$\tilde{F}(z) = (Cz, ACz) = (z, C^* ACz).$$

The formulas for $\tilde{A}$ and $\tilde{F}$ are similar, and this is used in practice.

1.4 Practice: $A =$ const.

Given L_x take $F_A = (y, Ay)$. Bring it to the canonical form: $y = Cz,\ \tilde{F} = (z, Bz), B$ is a diagonal matrix with $0, \pm 1$ on the diagonal. Take a coordinate change $\xi = Jx,\ J = C^*$. This change will take L to $\tilde{L},\ \tilde{L} = (\nabla_\xi, B\nabla_\xi)$.

Classification:

$B =$ diag $(1, ..., 1) \rightsquigarrow$ elliptic case;

$B =$ diag $(-1, 1, ..., 1) \rightsquigarrow$ hyperbolic case;

$B =$ diag $(0, 1, ..., 1) \rightsquigarrow$ elliptic case.

This is not a complete classification; we do not name here other cases.

1.5 Preserving the Laplace operator by a linear map.

See Problem 2 from the List 5.

1.6 Case $n = 2$.

Formula with matrixes $A(x) \rightsquigarrow \tilde{A} = J(x)A(x)J^*(x)$ for $n = 2$ implies (check it!): if

$$L_x u = a u_{x^2} + 2b u_{xy} + c u_{y^2} + d u_x + f u_y,$$

$$(x, y) \rightsquigarrow (\xi, \eta),$$

then

$$L_\xi u = \tilde{a} u_{\xi^2} + 2\tilde{b} u_{\xi\eta} + \tilde{c} u_{\eta^2} + \tilde{d} u_\xi + \tilde{f} u_\eta,$$

where

$$\tilde{a} = a\xi_x^2 + 2b\xi_x\xi_y + c\xi^2 y$$

$$\tilde{c} = \text{ same for } \xi \rightsquigarrow \eta$$

$$\tilde{b} = a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y$$

In the hyperbolic case we wish: $\tilde{a} = 0 \Leftrightarrow L_v\xi = 0$, or $L_w\xi = 0$, $v = (a, b + \sqrt{b^2 - ac}),\ w = (a, b - \sqrt{b^2 - ac})$. These are the characteristic equations, the orbits of v and w are called characteristics. Canonical form: $u_{\xi\eta} +$ l.o.t. $= 0$.

First order coefficients: $\tilde{d} = L\xi, \tilde{f} = \tilde{\eta}$.

1.7 General solution for $u_{\xi\eta} = 0$.

$$u = f(\xi) + g(\eta)$$

.

1.8 General solution for $u_{t^2} = a^2 u_{x^2}$.

See Problems 7, 8 from List 5.