Task 3: complex dynamics. Deadline: March 14

February 28, 2017

The Riemann–Hurwitz Formula. *Let $\pi : S_1 \to S_2$ be a branched covering of Riemann surfaces, where S_1 and S_2 are either both compact, or both compact with boundary, and let $A_1, \dots, A_k \in S_1$ be its critical points. We consider that they do not lie in the boundary of the surface S_1. For every A_j the germ of the projection $\pi : (S_1, A_j) \to (S_2, \pi(A_j))$ is a holomorphic mapping of the type $z \mapsto c_j z^{b_j+1} + o(z^{b_j+1})$, $c_j \neq 0$, $b_j \in \mathbb{N}$ in local coordinates centered at A_j and $\pi(A_j)$. The* **branching order** *of the covering π at A_j equals the number b_j: the* **local degree** *of the germ (the number of preimages of a point distinct from $\pi(A_j)$)* **minus one.** *Set*

$$\chi(S_i) = \text{ the Euler characteristic of the surface } S_i,\ i = 1, 2;\ d = \text{ the degree of the covering } \pi :$$

the number of preimages of a point distinct from the critical values $\pi(A_j)$ (it is well-defined and finite by compactness). Then

$$\chi(S_1) = d\chi(S_2) - \sum_{j=1}^{k} b_j. \tag{1}$$

Problem 1. Prove the Riemann–Hurwitz Formula (1).

Hint. Calculate the Euler characteristics of the surfaces S_i using their appropriate triangulations T_i. The critical values should be contained in the set of vertices of the triangulation T_2, and the triangulation T_1 should be its pullback.

Problem 2. Show that there exist no branched covering $\pi : S_1 \to S_2$ where

a) S_1 is simply connected and S_2 isn't;

b) S_1, S_2 are finitely-connected domains in the Riemann sphere and the number of holes in S_2 is bigger than that in S_1;

c) S_1, S_2 are compact Riemann surfaces and the genus of the source S_1 is less than the genus of the image S_2.

Problem 3. Let S_1, S_2 be either finitely-connected domains in the Riemann sphere with the same number of holes, or compact Riemann surfaces of the same *positive* genus. Then every branched covering $S_1 \to S_2$ (if any) is a conformal isomorphism.

Problem 4. Prove that each connected component of the Fatou set is either simply-connected, or 2-connected (has one hole) and is conformally-equivalent to an annulus, or infinitely-connected (has infinite number of holes).

Problem 5. Prove that the number of components of the Fatou set may take only one of the three following values: 0, 1, 2, ∞.

Problem 6. Prove that each connected component of the Fatou set of a polynomial that is different from the attracting basin of infinity is simply-connected: in particular a polynomial cannot have Herman rings.

Problem 7. Prove that the mapping $z \mapsto z^2 - z$ sends the disk $D_{\frac{1}{2}}$ conformally onto the main cardioid: the component of the Mandelbrot set that corresponds to the quadratic polynomials $z^2 + c$ having a (super) attracting fixed point. Deduce that the multiplier of the fixed point (considered as a function of c) conformally parametrizes the main cardioid by the unit disk.