# Task 1: almost complex structures. Deadline: February, 10

January 27, 2017

**Problem 1.** For every $\mathbb{R}$-linear operator $\mathbb{R}^2 \to \mathbb{C}$ given below find its $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts, i.e., express it as $A(z) = az + b\bar{z}$:

a) $A(x, y) = (x + 2y, 0)$;

b) $A(x, y) = (x, 2y)$;

c) $A(x, y) = (x + y, y)$;

d) $A(x, y) = (2x + y, x + y)$;

e) $A(x, y) = (2y, -x)$;

f) Represent the above operators from b)–e) as $A(z) = \lambda(z + \mu\bar{z})$ and find $\mu$.

**Problem 2.** Consider the standard linear complex structure $\sigma_{st}$ on $\mathbb{C}$. For the above operators b)–e) find the anti-involution $J = J_A$ defining the pullback linear complex structure

$$\sigma = \sigma_A = A_*^{-1}\sigma_{st}.$$

**Problem 3.** Find the ellipses defining the linear complex structures $\sigma_A$ for the operators $A$ from b)–e).

**Problem 4.** Find the dilatations of the above operators b)–e).

**Problem 5.** * Let $\sigma_1$ and $\sigma_2$ be two linear complex structures on $\mathbb{R}^2 \simeq \mathbb{C}$ respecting orientation. Let $\mu_j \in \mathbb{C}$, $|\mu_j| < 1$, $j = 1, 2$ be the complex numbers defining $\sigma_j$, i.e., $z \mapsto w_j = z + \mu_j\bar{z}$ is an $\mathbb{R}$-linear operator transforming $\sigma_j$ to the standard complex structure $\sigma_{st}$. Define the *dilatation* $K$ of the structure $\sigma_2$ with respect to $\sigma_1$ to be its dilatation written in the complex coordinate $w_1$ that is $\mathbb{C}$-linear for the structure $\sigma_1$. Prove that $\ln K$ equals the distance $dist_P(\mu_1, \mu_2)$ in the Poincaré metric of the unit disk $D_1$: the pullback of the Lobachevsky–Poincaré metric $\frac{dx^2+dy^2}{y^2}$ on the upper half-plane $\mathbb{H} = \{Imz > 0\}$ under a conformal isomorphism $D_1 \simeq \mathbb{H}$.

**Problem 6.** Find the image $F_*\sigma_{st}$ of the standard complex structure $\sigma_{st}$ on $\mathbb{C}$ under the following diffeomorphisms $F : \mathbb{C} \to \mathbb{C}$: more precisely, find the corresponding ellipse fields, $\mathbb{C}$-linear differentials, and functions $\mu$.

a) $F(x, y) = (x, g(y)$, where $g : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism;

b) $F(x, y) = (f(x), g(y))$, where $f, g : \mathbb{R} \to \mathbb{R}$ are diffeomorphisms;

c) $F(x, y) = (x + f(y), g(y))$, where $g : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism and $f : \mathbb{R} \to \mathbb{R}$ is smooth;

d) $F(r, \phi) = (r, \phi + \psi(r))$, where $\psi(r)$ is smooth as a function in $r^2$.

**Problem 7.** * Consider an almost complex structure $\sigma$ on $\mathbb{C}$ given by a function $\mu$ depending only on the coordinate $y$: $\mu = \mu(y)$. Find a diffeomorphism $F : \mathbb{C}^2 \to \mathbb{C}^2$ transforming $\sigma$ to the standard complex structure.