Task 2: quasiconformal mappings, Poincaré metrics and complex dynamics. Deadline: February, 24

February 10, 2017

Problem 1. Which mappings $\mathbb{R}^2 \to \mathbb{R}^2$ from the list are quasiconformal?
a) $(x, y) \mapsto (x, y^a \text{sign}(y))$, $a > 0$, $a \neq 1$;
b) $(x, y) \mapsto (x, y + g(x))$, $g \in C^1$, $0 < c_1 < |g'(x)| < c_2 < \infty$;
c) $(x, y) \mapsto (x, y + P(x))$, P is a polynomial;
d) $(r, \phi) \mapsto (r^a, \phi)$, $a > 0$, $a \neq 1$; (r, ϕ) are polar coordinates;
e) $(r, \phi)(\mapsto (r^a(1 + r^b), \phi + \theta)$, $a, b > 0$, $\theta \in \mathbb{R}$.
Explain your answer.

Consider Lobachevsky–Poincaré metric on the upper half-plane $\mathbb{H} = \{\operatorname{Im} z > 0\} \subset \mathbb{C}$:

$$g_P = \frac{|dz|}{|\operatorname{Im} z|} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

It is well-known that the metric g_P is invariant under the conformal automorphisms of the half-plane. Recall that a Riemann surface is *hyperbolic,* if its universal covering equipped with the pullback complex structure is conformally equivalent to $\mathbb{H}$ (or equivalently, to the unit disk). The *Poincaré metric* on a hyperbolic Riemann surface is the pushforward of the Poincaré metric of the universal covering half-plane under the covering projection.

Problem 2. Show that the above Poincaré metric is well-defined. Prove that in the standard coordinate z on D_1 the Poincaré metric of the unit disk is

$$g_P = \frac{2|dz|}{1 - |z|^2}.$$

Problem 3. Calculate the Poincaré metrics of the following domains:
a) Punctured unit disk $D_1 \setminus \{0\}$;
b) the strip $0 < \operatorname{Im} z < 1$;
c) an annulus $\{r < |z| < 1\}$.

Problem 4. Prove the **Schwarz Lemma in the invariant form** for holomorphic mappings $f : S_1 \to S_2$ of hyperbolic Riemann surfaces:

a) one has $|df(x)| \leq 1$ for every $x \in S_1$, here the norm is taken in the Poincaré metrics in the source and in the image; hence $dist(f(x), f(y)) \leq dist(x, y)$ for every $x, y \in S_1$;

b) if $|df(x)| = 1$ for some $x \in S_1$, then the mapping f is a non-ramified covering, and in this case it is a local isometry everywhere: $|df(x)| = 1$ for all $x \in S_1$.

Problem 5. * Prove Montel's Theorem: every family of holomorphic mappings $f_s : U \to \overline{\mathbb{C}}$ avoiding the same three fixed distinct values a, b and c is normal.

Hint. Use hyperbolicity of the triple punctured Riemann sphere. Consider its Poincaré metric, which is the multiple of the usual spherical metric by a continuous nonzero function. Use the above distance non-increasing property of the mappings f_s of a disk to $\overline{\mathbb{C}} \setminus \{a, b, c\}$. Deduce equicontinuity of the mappings f_s on compact sets.

Problem 6. * Prove normality of the family of normalized univalent holomorphic mappings

$$\mathcal{S} = \{f : D_1 \to \mathbb{C} \mid f \text{ is holomorphic and injective, } f(0) = 0,\ f'(0) = 1\}.$$

Hint. Use Montel's Theorem and Schwarz Lemma.

Problem 7. Prove that the Julia set of every Tchebyshev polynomial is the segment $[-1, 1]$.