1 Quasilinear partial differential equations

The main part of the lecture is covered by Section 7 of Arnold's "Geometric methods...".

1.1 Quasilinear equation

$$a_1(x,u)u_{x_1} + \cdots + a_n(x,u)u_{x_n} = b(x,u)$$

$$a(x,u)D_u = b(x,u) \tag{1}$$

Definition 1 $\mathbb{R}^{n+1} = \{(x,u)\}$

Characteristics in $\mathbb{R}^{n+1}$:

$$\begin{aligned} \dot{x} &= a \\ \dot{u} &= b \end{aligned} \tag{2}$$

Lemma 1 *Characteristics are tangent to the graph of any solution of* (1).

Proof If $u = f(x)$ is a solution of (1), then $b = aDf$. Let $F(x,u) = U - f(x),\ (a,b) = v$ Then $L_vF = b - aDf|_{u=f(x)} = 0$, as f is a solution. □

1.2 Invariant manifolds

Lemma 2 *If a manifold is tangent to a vector field, then it is saturated by its orbits.*

Proof A manifold has the form $F = 0$. Change coordinates to (x,y) so that it will be $y = 0 : \Pi$. The vector field tangent to Π has 0 y-component: $v(x,0) = (w(x),0)$. The phase curves of $\dot{x} = w(x),\ x \in \Pi$, are in Π. □

Corollary 1 *The graph of a solution of* (1) *is saturated by characteristics* (2).

1.3 Cauchy problem for (1)

A hypersurface $\gamma \subset \mathbb{R}^n$ is an initial surface, $\varphi : \Gamma \to \Gamma$ is an initial function.

$$u|_\gamma = \varphi \tag{3}$$

Remark 1 *If solution of* (1) *exists, then its graph is the saturation by characteristics of the graph of the initial function.*

This saturation must be a graph of a function.

Definition 2 *A point $x \in \gamma$ is characteristic iff $a(x, \varphi(x))$ is tangent to γ (belongs to $T_x\gamma$).*

Theorem 1 *Problem* (1), (3) *has one and only one solution near any non-characteristic point.*

Proof The saturation in the remark is a manifold. Projection of its tangent plane at $(x, \varphi(x))$ along the u-axis is $T_x\mathbb{R}^n$, because $a(x, \varphi(x)) \notin T_x\gamma$.

$\square$

Some words were said about shock waves and Euler equation.