1. An invariant of two-dimensional knots in four-space

A classical knot is an embedded circle in 3-space, and much of low-dimensional topology is concerned with the problem of deciding when two knots are the same. A natural measure of the complexity of a knot its *crossing number*: a generic projection of a knot onto the 2-dimensional plane is a circle with a number of self-intersections, where two arcs intersect at a point (called a double point); the crossing number is the minimal number of double points in any projection of that knot. This knot *invariant* (it does not change under isotopy of the knot) is used, for example, to organize different knots into *knot tables*.

A two-dimensional knot, or *2-knot*, is an embedded 2-sphere in 4-space. Now, a generic projection of a 2-knot onto 3-dimensional space may have self-intersections of two types: either two sheets intersect in a 1-dimensional curve of double points, or three sheets intersect at an isolated *triple point*. In analogy with the crossing number of a knot, the *triple point number* of a 2-knot is the minimal number of triple points in any 3-dimensional projection of that 2-knot. A 2-knot with vanishing triple point number is called *pseudo-ribbon*.

Not a lot is known about the triple point number of 2-knots, however, and consequently an organizational framework of 2-knots seems a long way off. Indeed, until 2004 it was not known if 2-knots which are not pseudo-ribbon even existed.

Recently, an invariant μ of 2-knots was proposed in [2] (see below) which obstructs a 2-knot being pseudo-ribbon. That is, if a 2-knot K is pseudo-ribbon, then $\mu(K) = 0$. (Thus, if $\mu(K) \neq 0$ then we know K is not pseudo-ribbon.) However, the only examples with $\mu \neq 0$ computed thus far were already known not to be pseudo-ribbon by other methods. The question then arises: can the invariant μ detect new examples of 2-knots which are not pseudo-ribbon?

The goal of this project is to address this question. In doing so you will learn about (among other things): 2-dimensional analogues of classical knot diagrams, what a *quandle* is and how this can be used to *color* a 2-knot, and how a certain homology may be associated to such a coloring to detect interesting 2-knots.

Pre-requisites: first courses in differential topology (in particular, transversality) and algebraic topology (homology, CW-complexes, fundamental group). A familiarity with knot theory is not necessary.

Suggested reading:

1. S. Carter, S. Kamada, and M. Saito, *Surfaces in 4-space*, Encyclopaedia of Mathematical Sciences, vol. 142, Springer-Verlag, Berlin, 2004. Low-Dimensional Topology, III.

2. T. Yashiro, *Pseudo-cycles of surface-knots*, J. Knot Theory Ramifications 25 (2016), no. 13, 1650068, 18 pp.

2. Vassiliev invariants and two-dimensional braids

Two of the most useful results in low-dimensional topology are the classical theorems of Alexander and Markov, which state that any link in 3-space (i.e., an embedding of a disjoint union of circles into $\mathbb{R}^3$) is represented by a closed *braid*, and braid representations of isotopic links are related by some elementary operations called *Markov moves*. These theorems enable braid theory to play an important role in knot theory; for example, the original formulation of the Jones polynomial was in terms of braids.

A (geometric) braid is the disjoint union of simple arcs in the solid cylinder $D^2 \times [0, 1]$ such that each arc ascends monotonically from $D^2 \times \{0\}$ to $D^2 \times \{1\}$. By taking a "movie" of braids (where we regard "time" as a fourth dimension) one may construct a 2-dimensional braid, or *2-braid*, in 4-space, and the Alexander and Markov theorems have natural analogues in this 4-dimensional setting: any (possibly disconnected) surface in 4-space may be represented by a 2-braid, and 2-braids representing isotopic surfaces are related by a sequence of Markov-like moves. The notion of 2-braids may further be generalized to *singular 2-braids*, where the underlying surface may have transverse self-intersections (which is the generic situation), and the theory of *Vassiliev invariants* has been adapted to this setting. In this project you will investigate the properties of these invariants.

In the classical setting, a Vassiliev invariant is a numerical invariant of (oriented) links which, when extended to singular links in a canonical way, satisfies a certain finiteness condition. A fundamental open question of low-dimensional topology is whether or not Vassiliev invariants distinguish all knots.

A numerical invariant of 2-braids in 4-space may be extended to singular 2-braids in a similar way. These Vassiliev-type invariants have been shown in [1] (see below) to depend on certain simple data associated to a 2-braid. The goal of this project is to adapt the methods of [1] to prove the following conjecture: any Vassiliev-type invariant of 2-links (i.e., an embedding of a disjoint union of 2-spheres into the 4-sphere) is *trivial*, meaning that the invariant has the same value for all 2-links. The conjecture is known to hold for 2-knots (one-component 2-links).

In doing so you will learn about (among other things): classical Vassiliev knot invariants and their generalizations, how to separate surfaces in 4-space, two-dimensional braids.

Pre-requisites: a first course in differential topology (in particular, transversality). A familiarity with knot theory is not necessary.

Suggested reading:

1. M. Iwakiri, *Unknotting singular surface braids by crossing changes.* Osaka Journal of Mathematics 45.1 (2008): 61-84.

2. S. Kamada, *Braid and knot theory in dimension four*. Vol. 95. American Mathematical Soc., 2002.