Task 4: complex dynamics and holomorphic motions. Deadline: April 7

March 17, 2017

Problem 1. Extend the following holomorphic motions of subsets $X \subset \overline{\mathbb{C}}$ (collections of disjoint graphs of holomorphic functions $\phi_z : D_1 \to \overline{\mathbb{C}}$, $\phi_z(0) = z$ for $z \in X$) to holomorphic motions of the whole Riemann sphere over the unit disk. That is, to filling of the whole product $D_1 \times \overline{\mathbb{C}}$ by disjoint graphs of holomorphic functions. Here for every $t \in D_1$ we denote

$$X_t = \cup_{z \in X} \phi_z(t) \subset \overline{\mathbb{C}}.$$

a) $X = \{0, 1, 2i, \infty\}$, $\phi_\infty(t) \equiv \infty$, $\phi_0(t) \equiv 0$, $\phi_1(t) \equiv 1$, $\phi_{2i}(t) = 2i + t$;

b) $X = \{A_1, \dots, A_n\}$ such that for every $t \in D_1$ the set X_t consists of vertices of a convex polygon containing zero in its interior;

c) any holomorphic motion of a subset $X \subset \overline{\mathbb{C}}$ such that for every $t \in D_1$ the complement $\overline{\mathbb{C}} \setminus X_t$ is a convex subset in $\mathbb{C}$.

Problem 2. Are the following rational functions J-stable? Are they structurally stable? Are they hyperbolic?

a) $f(z) = z^2 + \lambda z$, $0 < |\lambda| < 1$;

b) $f(z) = z^2 + \lambda z$, $|\lambda| = 1$;

c) $f(z) = z^2 + \lambda z$, $|2 - \lambda| < 1$;

d) $f(z) = z^3 - \frac{1}{4}z$;

e) $f(z) = z + \frac{1}{z}$.

Problem 3. For those above functions that are hyperbolic find the corresponding moduli spaces as unions of punctured Riemann surfaces.

Problem 4. A *finite Blaschke product* is a product

$$\prod_{j=1}^{n} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a}_j z}.$$

Show that if $|a_j| < 1$ for all j, then the Julia set of the corresponding Blaschke product is the unit circle, and the unit disk and its exterior are completely invariant, i.e., forward and backward invariant subsets.

Problem 5. ** A Lattès example is a rational function coming from an expanding torus endomorphism (e.g., torus doubling map $z \mapsto 2z$) passing to the quotient of the torus by the central symmetry $z \mapsto -z$. (The latter quotient equipped with the natural complex structure is conformally equivalent to the Riemann sphere.) Find an explicit rational function realizing the Lattès example for the torus doubling map.

Hints. 1) Use the fact that each complex torus is realized as an elliptic curve: an algebraic curve

$$E = \{y^2 = P_3(x)\} \subset \mathbb{CP}^2, \ P_3 \text{ is a cubic polynomial with distinct roots.}$$

2) Use the geometric description of the additive group structure on the elliptic curve: for every three points $A, B, C \in E$ one has $A + B + C = 0$, if and only if they lie on the same line; zero can be taken to be any inflection point, e.g., the point at infinity (show that it is indeed an inflection point).

3) Deduce that the projection to the quotient Riemann sphere corresponds to the projection $(x, y) \mapsto x$.

4) Using the above geometric addition law and the fact that the point at infinity can be taken as zero, find geometric representation of the doubling map. Deduce the formula for the quotient of the doubling map.