Task 5: Kleinian groups. Deadline: April 21

April 6, 2017

Problem 1. Prove that each automorphism of the Riemann sphere is conformally conjugated to either $z \mapsto \lambda z$, $\lambda \neq 0$, or $z \mapsto z + 1$. Deduce the classification of conformal automorphisms of the Riemann sphere up to conformal conjugacy as was presented during the course:
- loxodromic: $\lambda \notin \mathbb{R} \cup \{|z = 1\})$;
- hyperbolic: $\lambda \in \mathbb{R} \setminus \{|z| = 1\}$;
- elliptic: $|\lambda| = 1$;
- parabolic: $z \mapsto z + 1$.

Problem 2. Prove that if a conformal automorphism of the Riemann sphere is not parabolic, then the multipliers at its two fixed points are inverse to each other.

Problem 3. a) Let $a, b \in Aut(\overline{\mathbb{C}})$, and let a be not elliptic. Prove that a and b commute, if and only if they have common fixed points.

b) Classify all the commuting pairs (a, b) up to simultaneous conformal conjugacy.

Problem 4. a) Prove that the commutator of any two automorphisms of the Riemann sphere having a common fixed point is either identity, or a parabolic transformation.

b) Prove that any two non-commuting elements of $Aut(\overline{\mathbb{C}})$ having a common fixed point generate a solvable non-discrete subgroup.

c)* Prove that every non-commutative discrete subgroup in $Aut(\overline{\mathbb{C}})$ containing at least one non-elliptic element is non-solvable.

d)* Find an example of a non-commutative *solvable* discrete subgroup in $Aut(\overline{\mathbb{C}})$.

Problem 5. * Consider a Fuchsian group $\Gamma \subset Aut(D_1)$ whose quotient D_1/Γ is once punctured complex torus. How many sides may have the corresponding Dirichlet fundamental polygon?

Problem 6. a) Consider the standard fundamental domain in $\mathbb{H} = \{Imz > 0\}$ for the modular group $PSL_2(\mathbb{R}) = Aut(\mathbb{H})$: the domain bounded by two vertical lines with abscissas $\pm\frac{1}{2}$ and and arc of the unit circle centered at $0 \in \partial\mathbb{H}$. Prove that is has a finite area in the Lobachevsky metric $\frac{dx^2+dy^2}{y^2}$.

b) Extend this result to any geodesic polygon in $D_1 \simeq \mathbb{H}$ with finite number of sides and without absolute sides (sides in ∂D_1).

C) Let $\Gamma \subset Aut(D_1)$ be a finitely generated discrete torsion free subgroup with $\Lambda = \partial D_1$. Prove that the corresponding Dirichlet fundamental polygon has finite area. This implies that *every hyperbolic Riemann surface of finite type has a finite area in its Poincaré metric induced by the Lobachevsky–Poincaré metric of its universal covering $D_1 \simeq \mathbb{H}$.*

Problem 7. a)* Prove that a fundamental polygon of a discrete subgroup $\Gamma \subset Aut(D_1)$ has a finite volume, if and only if it has a finite number of sides and no absolute sides.

Hint. Use Gauss-Bonnet formula for parallel transform of a vector along geodesic sides.

b) Deduce that a hyperbolic Riemann surface has a finite volume, if and only if it is of finite type.