Simple proofs of uniformization theorems

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Abstract

The measurable Riemann mapping theorem proved by Morrey and in some particular cases by Ahlfors, Lavrentiev and Vekua, says that any measurable almost complex structure on $\mathbb{R}^2$ (S^2) with bounded dilatation is integrable: there is a quasiconformal homeomorphism of $\mathbb{R}^2$ (S^2) onto $\mathbb{C}$ ($\overline{\mathbb{C}}$) transforming the given almost complex structure to the standard one. We give an elementary proof of this theorem.

First we prove its double-periodic version: each C^∞ almost complex structure on the two-torus can be transformed by a diffeomorphism to the standard complex structure on appropriate complex torus. The proof is based on the homotopy method for the Beltrami equation on $\mathbb{T}^2$ with a parameter. As a by-product, we present a simple proof of the Poincaré-Köbe theorem saying that each simply-connected Riemann surface is conformally equivalent to either $\overline{\mathbb{C}}$, or $\mathbb{C}$, or the unit disc. We prove this theorem modulo the purely topological statement that any noncompact simply-connected Riemann surface is C^∞- diffeomorphic to plane. Afterwards the general case of the measurable Riemann mapping theorem is treated by C^∞ double-periodic approximation and simple normality arguments (involving Grötzsch inequality) following the classical scheme.

Contents

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1 Introduction, the plan of the paper and history

1.1 Uniformization theorems. The plan of the paper

A linear complex structure on $\mathbb{R}^2$ is a structure of a linear space over $\mathbb{C}$. We fix an orientation on $\mathbb{R}^2$ and consider it to be compatible with the complex structure. An *(almost) complex structure* on a real two-dimensional surface is a family of linear complex structures on the tangent planes at its points. A linear complex structure on $\mathbb{R}^2$ defines an ellipse in $\mathbb{R}^2$ centered at 0, which is an orbit under the S^1- action by multiplication by complex numbers with unit modulus. (This ellipse is unique up to a homothety. The ellipse corresponding to the standard complex structure on $\mathbb{C}$ is a circle.) The *dilatation* of a nonstandard linear complex structure on $\mathbb{C}$ (with respect to the standard complex structure) is the *aspect ratio* of the corresponding ellipse. This is the ratio of the largest radius over the smallest one. An almost complex structure defines an ellipse field in the tangent planes, and vice versa: an ellipse field determines the almost complex structure in the unique way.

If our surface is a Riemann surface (with a fixed complex structure), then any (nonstandard) almost complex structure has a well-defined dilatation at each point of the surface. The *(total) dilatation* of an almost complex structure is the essential supremum of its dilatations at all the points. This is the minimal supremum of the dilatations after possible correction of the almost complex structure over a measure zero set. An almost complex structure is said to be *bounded*, if its total dilatation is finite.

Each real linear isomorphism $\mathbb{C} \to \mathbb{C}$ acts on the space of the ellipses centered at 0, and hence, on the space of linear complex structures. Its *dilatation* is defined to be the dilatation of the image of the standard complex structure. It is equal to the aspect ratio of the image of a circle centered at 0. The action of a differentiable homeomorphism of domains in $\mathbb{C}$ on the almost complex structures and its dilatation (at a point) are defined to be those of its derivative (at the points where the derivative exists and is a nondegenerate linear operator). At those points where the derivative exists and is a nonzero degenerate operator, the dilatation is defined to be infinite. The *(total)* dilatation is the essential supremum of the dilatations through all the previous points.

Any C^∞ (and even measurable) bounded almost complex structure is integrable, that is, can be transformed to a true complex structure by a C^∞ (respectively, quasiconformal) homeomorphism, see the following Definition and Theorem.

1.1 Definition (see [Ah2]). Let $K > 0$. A diffeomorphism of domains in $\mathbb{C}$ is said to be a *K-diffeomorphism*, if its dilatation is no greater than K. A homeomorphism of domains in $\mathbb{C}$ is said to be K- *quasiconformal* (or K-homeomorphism), if it has local L_2 distributional derivatives and its total dilatation is no greater than K. A homeomorphism is said to be quasiconformal if it is K- quasiconformal for some $K > 0$.

1.2 Remark The dilatations of a two-dimensional linear operator and of its inverse are equal. The inverse to a K-diffeomorphism is also a K-diffeomorphism. The composition of two K-diffeomorphisms is a K^2-diffeomorphism. This follows from the definition.

1.3 Proposition *(see [Ah2]) The quasiconformal homeomorphisms of a Riemann surface form a group.*

This proposition is proved in Subsection 3.4 for completeness of presentation, however it will not be used in the paper.

1.4 Definition A homeomorphism $\mathbb{C} \to \mathbb{C}$ is said to be *normalized*, if it fixes 0 and 1.

1.5 Theorem *(Existence, [AhB], [M]). For any measurable bounded almost complex structure σ on $\mathbb{C}$ there exists a unique normalized quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ that transforms σ to the standard complex structure (at almost any point outside the zero locus of its derivative). If σ is C^∞ in some domain, then the homeomorphism is a C^∞ diffeomorphism while restricted to this domain.*

1.6 Theorem *(Dependence on Parameters, [AhB]). If a bounded almost complex structure on $\mathbb{C}$ varies analytically in a complex parameter, then so does the corresponding homeomorphism from Theorem 1.5.*

1.7 Remark A quasiconformal homeomorphism of a once punctured domain extends quasiconformally to the puncture (in particular, the homeomorphism from Theorem 1.5 is quasiconformal at infinity). This follows easily from the local uniqueness of the quasiconformal homeomorphism up to composition with conformal mapping (Proposition 3.12, see Subsection 3.4) and the theorem on erasing isolated singularities of bounded holomorphic functions.

We give proofs of Theorem 1.5 in Sections 2 and 3 and of Theorem 1.6 in Subsection 3.5 that seem to be simpler than the known proofs, a historical overview of which is provided in Subsection 1.4.

1.8 Remark The proof of the local integrability of an analytic almost complex structure is elementary and due to Gauss. It is done immediately by analyzing the complexification of the corresponding $\mathbb{C}$- linear 1- form. But the proof is already nontrivial in the C^∞ case.

The measurable versions of Theorems 1.5 and 1.6 have many very important applications in various domains of mathematics, especially in holomorphic dynamics and the theory of Kleinian groups. More specifically, in holomorphic dynamics a technique called quasiconformal surgery depends upon invariant almost complex structures that are discontinuous, see, for example [CG].

In order to prove Theorem 1.5, in Section 2 we first prove its version for C^∞ almost complex structures on the two-torus. The proof only uses the elementary Fourier analysis.

1.9 Theorem *([Ab]) For any C^∞ almost complex structure σ on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ (which is always bounded) there exists a C^∞ diffeomorphism of $\mathbb{T}^2$ onto an appropriate complex torus, depending on σ, that transforms σ to the standard complex structure.*

In Section 3 we then deduce Theorem 1.5 from Theorem 1.9 by using double-periodic approximations of a given almost complex structure on $\mathbb{C}$ and simple normality arguments involving a Grötzsch inequality for annuli diffeomorphisms. This deduction follows the classical scheme [Ah2].

The proof of Theorem 1.9 presented below is implicitly contained in a previous paper by the author [Gl], where the same method was used to prove a foliated version of Theorem 1.5. We prove the existence of a global nowhere vanishing σ- holomorphic differential. To do this, we use the homotopy method for the Beltrami equation with a parameter, which reduces the proof to solving a linear ordinary differential equation in $L_2(\mathbb{T}^2)$. We prove regularity of its solution by showing that the equation is bounded in any Sobolev space $H^s(\mathbb{T}^2)$.

In Subsection 1.3 we give a proof of the classical Poincaré-Köbe uniformization theorem using Theorem 1.9 (modulo a topological statement on simply-connected Riemann surfaces):

1.10 Theorem *[Ko1], [Ko2], [P]. Each simply-connected Riemann surface is conformally equivalent to either the unit disc, $\mathbb{C}$, or the Riemann sphere.*

In the proofs of the previously mentioned theorems we use the well-known notations (recalled in the next subsection) concerning almost complex structures.

1.2 Complex structures and uniformizing differentials: basic notations

To a nonstandard almost complex structure, denoted by σ, on a subset $D \subset \mathbb{C}$ we put into correspondence a $\mathbb{C}$- valued 1- form that is $\mathbb{C}$- linear with respect to σ. The latter form can be normalized to be

$$\omega_\mu = dz + \mu(z)d\bar{z}, \ \ |\mu| < 1. \tag{1.1}$$

The function μ is uniquely defined by σ. Conversely, for an arbitrary complex-valued function μ, with $|\mu| < 1$, the 1- form ω_μ defines the unique complex structure for which ω_μ is $\mathbb{C}$- linear. We denote by σ_μ the almost complex structure thus defined (whenever the contrary is not specified). Then σ_μ is bounded, if and only if the essential supremum of the function $|\mu|$ is less than 1.

1.11 Remark The ellipse associated to σ_μ on the tangent plane at a point z is given by the equation $|dz + \mu(z)d\bar{z}| = 1$; the dilatation (aspect ratio) is equal to $\frac{1+|\mu(z)|}{1-|\mu(z)|}$.

We will be looking for a differentiable homeomorphism $\Phi(z)$ that is holomorphic; it transforms σ_μ to the standard complex structure. This is equivalent to say that the differential of Φ (which is a closed form) is a $\mathbb{C}$- linear form, or equivalently, has the type $f(z)(dz + \mu d\bar{z})$:

$$\frac{\partial\Phi}{\partial\bar{z}} = \mu\frac{\partial\Phi}{\partial z} \text{ (Beltrami equation).} \tag{1.2}$$

1.12 Remark Conversely, let μ be C^∞, with $|\mu| < 1$. Then any C^∞ closed 1- form $f(z)(dz+\mu d\bar{z})$ is σ_μ- holomorphic, i.e., is locally a differential of a complex-valued C^∞ function Φ

transforming σ_μ to the standard complex structure. A form $f(z)(dz + \mu d\bar{z})$ is closed if and only if

$$\partial_{\bar{z}} f = \partial_z(\mu f). \tag{1.3}$$

1.13 Definition A Riemann surface is said to be *parabolic*, if its universal covering is conformally equivalent to $\mathbb{C}$ (In this case one can check that the surface must be either $\mathbb{C}$, $\mathbb{C}^*$, or a complex torus.)

1.14 Definition The *uniformizing differential* on $\mathbb{C}$ (or on a complex torus) with the affine coordinate z is the 1- form dz or a nonzero constant multiple of dz. More generally, a holomorphic 1- form on a parabolic Riemann surface is said to be *a uniformizing differential*, if its lifting to the universal cover is the differential of a conformal isomorphism onto $\mathbb{C}$.

1.15 Remark The uniformizing differential is well-defined up to multiplication by constant. It coincides with the unique (up to constant) nowhere vanishing holomorphic differential whose squared modulus is a complete metric.

1.16 Proposition *Let $\mu : \mathbb{T}^2 \to \mathbb{C}$ be a C^∞ function with $|\mu| < 1$. Suppose there is a C^∞ nowhere vanishing function $f : \mathbb{T}^2 \to \mathbb{C} \setminus 0$ satisfying (1.3). Then the corresponding almost complex structure σ_μ is integrable and the form $f\omega_\mu$ is a uniformizing differential of $(\mathbb{T}^2, \sigma_\mu)$.*

The proposition follows from compactness and the two previous remarks.

1.17 Remark A homeomorphism of domains in $\mathbb{C}$ with local L_2 derivatives is quasiconformal, if and only if it transforms some bounded measurable almost complex structure to the standard one (at almost any point outside the zero locus of its derivative); or equivalently, satisfies Beltrami Equation (1.2) almost everywhere with some measurable function μ having modulus with essential supremum less than one.

1.3 Proof of the uniformization Theorem 1.10 modulo Theorem 1.9 and a topological statement

1.18 Theorem *Any simply-connected Riemann surface is C^∞- diffeomorphic to either plane or sphere.*

Theorem 1.18 is a well-known (nontrivial) purely topological fact, and we will not prove it here. Theorem 1.18 is proved simultaneously with the uniformization theorem in [H], and its proof given there uses harmonic functions and complex analysis. To the knowledge of the author, this is one of the simplest known proofs of Theorem 1.18, and it is not known, whether there is a simple purely topological proof.

Let S be a simply-connected Riemann surface. Then it is diffeomorphic either to $\mathbb{R}^2$, or to the two-sphere (Theorem 1.18). We prove the statement of Theorem 1.10 first in the case, when S is diffeomorphic to $\mathbb{R}^2$. We show that S is conformally equivalent to either $\mathbb{C}$ or disc. Then if S is sphere, it follows that S is conformally equivalent to $\overline{\mathbb{C}}$ (by the previous statement applied to once punctured S and Riemann's Removable Singularity Theorem, which gives that an isolated singularity of a bounded holomorphic function is removable). In the proof of Theorem 1.10 we use the following corollary of Theorem 1.9 and the Riemann mapping theorem (that any simply-connected domain in $\mathbb{C}$ distinct from $\mathbb{C}$ is conformally equivalent to unit disc: the proof is elementary and is contained in standard courses of complex analysis.)

1.19 Corollary *For any C^∞ almost complex structure σ on the closed unit disc $\overline{D}$ there exists a C^∞ diffeomorphism of the open disc D onto itself transforming σ to the standard complex structure.*

Proof Let us extend σ to a double-periodic C^∞ almost complex structure on $\mathbb{R}^2$ (say, with periods 4 and $4i$) and consider the quotient torus equipped with the induced almost complex structure. Then the corresponding diffeomorphism for tori provided by Theorem 1.9 transforms the latter structure to the standard one. Its lifting to the universal covers transforms D to a simply-connected domain in $\mathbb{C}$ and sends σ to the standard complex structure. Now, applying the Riemann mapping theorem to the image of D, proves the corollary. □

By Theorem 1.19, there exists a C^∞- diffeomorphism of S onto $\mathbb{R}^2$. Denote by σ the image of the standard complex structure of S (this is a C^∞ almost complex structure on $\mathbb{R}^2$, but not necessarily bounded). Take a growing sequence of discs $S_1 \Subset S_2 \Subset \dots S = \mathbb{R}^2$ exhausting $\mathbb{R}^2$ centered at 0. By the corollary, for any n there is a diffeomorphism $\phi_n : S_n \to D$ conformal with respect to the complex structure of S, and with $\phi_n(0) = 0$. Let w be a local holomorphic chart on S near 0, also with $w(0) = 0$. Let us change ϕ_n by a constant multiple, obtaining $\Phi_n = \lambda_n \phi_n$ having unit derivative in w at 0. The family Φ_n is normal: each subsequence contains a subsequence converging uniformly on compact sets in S. Indeed, fix a $k \in \mathbb{N}$ and consider the C^∞ injections $\Phi_n \circ \phi_k^{-1} : D \to \Phi_n(S_k)$, $n \geq k$. By construction, the latter are holomorphic and univalent, they send 0 to 0 and have one and the same derivative at 0. Therefore, they form a normal family, see [CG], hence, so do the Φ_n's. By construction, the limit of a converging subsequence of the Φ_n' s is a conformal diffeomorphism of S onto either a disc, or $\mathbb{C}$. Theorem 1.10 is proved.

1.4 Historical overview

The local integrability of a C^∞ (and even Hölder) almost complex structure, as stated in Theorem 1.5, was proved by Korn [Korn] and Lichtenstein [Licht]; a simpler proof was obtained by Chern [Chern] and Bers [Be]. The local integrability together with the Poincaré-Köbe uniformization Theorem 1.10 imply the global integrability of C^∞ almost complex structure in Theorem 1.5. Lavrentiev [La] gave a direct proof of Theorem 1.5 for continuous almost complex structures. Later Ahlfors [Ah1] and Vekua [Vek] gave other direct proofs under the previous (stronger) Hölder condition.

In the general (measurable) case, Theorem 1.5 was proved by Morrey [M]. Later new proofs were obtained by Ahlfors and Bers [AhB], Bers and Nirenberg [BeN] and Boyarskii [Bo]. (In fact, Lavrentiev and Morrey stated their theorems for almost complex structures on a disc, but their versions on $\mathbb{R}^2$ follow immediately, by normality arguments similar to those from the previous subsection.) A simpler proof of Theorem 1.5 using L_2 analysis and Fourier transformation on $\mathbb{R}^2$ was obtained by A.Douady and X.Buff [DB] in 2000.

2 Smooth complex structures on $\mathbb{T}^2$. Proof of Theorem 1.9

2.1 Homotopy method. The sketch of the proof of Theorem 1.9

Let $\mu : \mathbb{T}^2 \to \mathbb{C}$ be a C^∞ complex-valued function with $|\mu| < 1$, and σ_μ be the corresponding almost complex structure, as in (1.1). Theorem 1.9 says that there exists a diffeomorphism transforming $(\mathbb{T}^2, \sigma_\mu)$ into a complex torus equipped with the standard complex structure. To prove this statement, it suffices to construct a uniformizing differential. More precisely, one constructs a C^∞ nowhere vanishing function $f : \mathbb{T}^2 \to \mathbb{C} \setminus 0$ such that the form $f\omega_\mu$ is closed (see Proposition 1.16). This corresponds to solving partial differential Equation (1.3) for a nowhere vanishing C^∞ function f.

To solve (1.3), we use the homotopy method. Namely, we include σ_μ into the one-parametric family of complex structures (denoted by σ_ν) defined by their $\mathbb{C}$- linear 1- forms

$$\omega_\nu = dz + \nu(z,t)d\bar{z}, \ \nu(z,t) = t\mu(z), \ t \in [0,1].$$

The complex structure corresponding to the parameter value $t = 0$ is the standard one, the given structure σ_μ corresponds to $t = 1$. We will find a C^∞ family $f(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C} \setminus 0$ of complex-valued nowhere vanishing C^∞ functions on $\mathbb{T}^2$ depending on the same parameter t, $f(z,0) \equiv 1$, such that the differential forms $f(z,t)\omega_\nu$ are closed, i.e.,

$$\partial_{\bar{z}} f = \partial_z (f\nu). \tag{2.1}$$

Then, the function $f = f(z,1)$ is the one we are looking for.

To construct the above-mentioned family of functions, we will find first a family $f(z,t)$ of *nonidentically-vanishing* (not necessarily nowhere vanishing) functions satisfying (2.1):

2.1 Lemma *Let $\nu(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ be a C^∞ family of C^∞ functions on $\mathbb{T}^2$, $|\nu| < 1$, $\nu(z,0) \equiv 0$, and z be the complex coordinate on $\mathbb{T}^2$. There exists a C^∞ family $f(z,t)$: $\mathbb{T}^2 \times [0,1] \to \mathbb{C}$ of C^∞ functions on $\mathbb{T}^2$ that are solutions of (2.1) with the initial condition $f(z,0) \equiv 1$ such that for any fixed $t \in [0,1]$ $f(z,t) \not\equiv 0$ in z.*

The lemma will be proved in the next subsection.

Below we show that, in fact, the functions $f(z,t)$ from the lemma vanish nowhere. To do this (and only in this place) we use the local integrability of a C^∞ complex structure:

2.2 Proposition *([Korn], [Licht], [La], [Chern], [Be]). Let $D \subset \mathbb{C}$ be a disc centered at 0, $\mu : D \to \mathbb{C}$, $\mu \in C^\infty$, $|\mu| < 1$, σ_μ be the corresponding almost complex structure, see (1.1). There exists a local σ_μ- holomorphic univalent coordinate near 0.*

The proposition will be proved in Subsection 2.3.

Proof of Theorem 1.9 modulo Lemma 2.1 and Proposition 2.2. Let $f(z,t)$ be a family of functions from Lemma 2.1. By the previous discussion, it suffices to show that $f(z,t) \neq 0$. This inequality holds for $t = 0$, where $f = 1$.

Let us prove that $f(z,t) \neq 0$ by contradiction. Suppose the contrary. Then the set of the parameter values t corresponding to the functions $f(z,t)$ having zeroes is nonempty (denote this set by M). Its complement $[0,1] \setminus M$ is open, because f is continuous. Let us show that the set M is open as well. This will imply that the parameter segment is a union of two disjoint nonempty open sets, which will bring us to contradiction. It is sufficient to show that the (local) presense of a zero of a function f persists under perturbation.

Suppose $f(z_0,t) = 0$ for some fixed z_0, t. It suffices to show that for t' close to t the function $f(z,t')$ has a zero near z_0. Let w be the local holomorphic coordinate on $\mathbb{T}^2$ near z_0 from Proposition 2.2 corresponding to $\mu = \nu(z,t)$, $w(z_0) = 0$. Suppose that the function $f(z,t)$ does not vanish identically on $\mathbb{T}^2$ locally near z_0: one can achieve this by changing z_0, since f does not vanish identically. Recall that $f\omega_\nu$ is a closed $\mathbb{C}$- linear 1-form with respect to the variable complex structure σ_ν, hence, it is holomorphic in the coordinate w. Therefore, $f\omega_\nu = (w^k + \text{higher terms})dw$, $k \geq 1$. Now by an index argument, the local presense of zero of f on $\mathbb{T}^2$ persists under perturbation. This together with the previous discussion proves the inequality $f(z,t) \neq 0$ and Theorem 1.9. □

2.2 Variable holomorphic differential: proof of Lemma 2.1

Differentiating (2.1) in t yields (we denote by $\dot{f}$ the partial derivative in t of a function f)

$$\partial_{\bar{z}}\dot{f} - (\partial_z \circ \nu)\dot{f} = (\partial_z \circ \dot{\nu})f. \tag{2.2}$$

where $\partial_z \circ \nu$ $(\partial_z \circ \dot{\nu})$ is the composition of the operator of the multiplication by the function ν (respectively, $\dot{\nu}$) and the operator ∂_z. Any solution f of Equation (2.2) with the initial condition $f(z,0) \equiv 1$ that does not vanish identically on the torus for any value of t is a one we are looking for. Let us show that (2.2) is implied by a bounded linear differential equation in $L_2(\mathbb{T}^2)$. To do this, we use the following properties of the operators ∂_z and $\partial_{\bar{z}}$.

2.3 Remark Denote $z = x_1 + ix_2$, $x = (x_1, x_2) \in \mathbb{R}^2$. The operators ∂_z, $\partial_{\bar{z}}$ on $\mathbb{T}^2$ have common eigenfunctions $e_n(x) = e^{i(n,x)}$, $n = (n_1, n_2) \in \mathbb{Z}^2$. The corresponding eigenvalues (denote them by λ_n and λ'_n respectively) have equal moduli, more precisely,

$$\lambda'_n = -\overline{\lambda_n}. \tag{2.3}$$

This is implied by the fact that the operator $\partial_{\bar{z}}$ is conjugated to $-\partial_z$ in the L_2 scalar product, which follows by definition. In fact,

$$\lambda_n = \frac{i}{2}(n_1 - in_2) \text{ and } \lambda'_n = \frac{i}{2}(n_1 + in_2).$$

2.4 Corollary *There exists a unique unitary operator $U : L_2(\mathbb{T}^2) \to L_2(\mathbb{T}^2)$ preserving averages such that "$U = \partial_{\bar{z}}^{-1} \circ \partial_z$" (more precisely, $U \circ \partial_{\bar{z}} = \partial_{\bar{z}} \circ U = \partial_z$). The operator U commutes with partial differentiations and extends to a unitary operator to any Hilbert Sobolev space of functions on $\mathbb{T}^2$. In particular, it preserves the space of C^∞ functions.*

Proof The operator U from the corollary is defined to have the eigenfunctions e_n with the eigenvalues $\frac{\lambda_n}{\lambda'_n} = \frac{n_1 - in_2}{n_1 + in_2}$. Its uniqueness follows immediately from the previous operator equation on U applied to the functions e_n. The rest of the statements of the corollary follow immediately from the definition and the Sobolev embedding theorem (see [Ch], p.411). □

Let us write down Equation (2.2) in terms of the new operator U. Applying the "operator" $\partial_{\bar{z}}^{-1}$ to (2.2) and substituting $U = \partial_{\bar{z}}^{-1} \circ \partial_z$ yields

$$(Id - U \circ \nu)\dot{f} = (U \circ \dot{\nu})f.$$

This equation implies (2.2). For any $t \in [0,1]$ the operator $Id - U \circ \nu$ in the left-hand side is invertible in $L_2(\mathbb{T}^2)$ and the norm of the inverse operator is bounded uniformly in t, since U is unitary and the modulus $|\nu|$ is less than 1 and bounded away from 1 by compactness. Thus, the last equation can be rewritten as

$$\dot{f} = (Id - U \circ \nu)^{-1}(U \circ \dot{\nu})f, \tag{2.4}$$

which is an ordinary differential equation in $f \in L_2(\mathbb{T}^2)$ with a uniformly L_2- bounded operator in the right-hand side. As it is shown below in Proposition 2.5, the inverse $(Id - U \circ \nu)^{-1}$ is also uniformly bounded in each Hilbert Sobolev space $H^j(\mathbb{T}^2)$. Therefore, Equation (2.4) written in arbitrary Hilbert Sobolev space has a unique solution with a given initial condition, in particular, with $f(z,0) \equiv 1$. (See the theorem on existence and uniqueness of solution of ordinary differential equation in Banach space with the right-hand side having uniformly bounded derivative [Ch].) For any $t \in [0,1]$ this solution does not vanish identically on $\mathbb{T}^2$ (uniqueness of solution) and belongs to all the spaces $H^j(\mathbb{T}^2)$; hence, it is $C^\infty(\mathbb{T}^2)$ by Sobolev embedding theorem (see [Ch], p.411). Thus, Lemma 2.1 is implied by the following

2.5 Proposition *Let $x = (x_1, x_2)$ be affine coordinates on $\mathbb{R}^2$, $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Let $s \geq 0$, $s \in \mathbb{Z}$, U be a linear operator in the space of C^∞ functions on $\mathbb{T}^2$ that commutes with the operators $\frac{\partial}{\partial x_i}$, $i = 1, 2$, and extends to any Sobolev space $H^j = H^j(\mathbb{T}^2)$, $0 \leq j \leq s$, up to a unitary operator. Let $0 < \delta < 1$, $\nu \in C^s(\mathbb{T}^2)$ be a complex-valued function, $|\nu| \leq \delta$.*

Then the operator $Id - U \circ \nu$ is invertible and the inverse operator is bounded in all the spaces H^j, $0 \leq j \leq s$. For any $0 < \delta < 1$, $j \leq s$, there exists a constant $C > 0$ (depending only on δ and s) such that for any complex-valued function $\nu \in C^s(\mathbb{T}^2)$ with $|\nu| \leq \delta$

$$||(Id - U \circ \nu)^{-1}||_{H^j} \leq C(1 + \sum_{k \leq j,\ i_r = 1,2} \max |\frac{\partial^k \nu}{\partial x_{i_1}, \dots, \partial x_{i_k}}|^j).$$

Proof Let us prove the proposition for $j = 0, 1$. For higher j its proof is analogous.

$$\text{By definition,} \ \ ||U \circ \nu||_{L_2} \leq \delta < 1. \tag{2.5}$$

Hence, the operator $Id - U \circ \nu$ is invertible in $L_2 = H^0$ and

$$(Id - U \circ \nu)^{-1} = Id + \sum_{k=1}^{\infty} (U \circ \nu)^k. \tag{2.6}$$

The sum of the L_2 operator norms of the sum entries in (2.6) is no greater than $\frac{\delta}{1-\delta}$ by (2.5). This proves the proposition for $j = 0$ and $C = \frac{1}{1-\delta}$. Let us prove it for $j = 1$. Let us show that the operator in the right-hand side of (2.6) is well-defined and bounded in H^1. To do this, it is sufficient to show that the sum of the operator H^1- norms of the same entries is finite.

Let $f \in H^1(\mathbb{T}^2)$. Let us estimate $||(U \circ \nu)^k f||_{H^1}$. We show that for any $k \in \mathbb{N}$

$$||\frac{\partial}{\partial x_r}((U \circ \nu)^k f)||_{L_2} \leq ck\delta^{k-1}||f||_{H^1} \text{ with } c = \delta + \max|\frac{\partial \nu}{\partial x_r}| \text{ for } r = 1, 2. \tag{2.7}$$

This will imply the finiteness of the operator H^1- norm of the sum in the right-hand side of (2.6) and Proposition 2.5. (Here $C = 4\sum_{k \in \mathbb{N}} k\delta^{k-1} = \frac{4}{(1-\delta)^2}$.)

Let us prove (2.7), e.g., for $r = 1$. The derivative in the left-hand side of (2.7) equals

$$(U \circ \nu)^k \frac{\partial f}{\partial x_1} + \sum_{i=1}^{k} (U \circ \nu)^{k-i} \circ (U \circ \frac{\partial \nu}{\partial x_1}) \circ (U \circ \nu)^{i-1} f$$

(since U commutes with the partial differentiation by the condition of Proposition 2.5). The L_2- norm of the first term in the previous formula is no greater than $\delta^k||f||_{H^1}$ by (2.5). Each term in its sum has L_2- norm no greater than $\delta^{k-1}\max|\frac{\partial\nu}{\partial x_1}|||f||_{L_2}$ by (2.5). This proves (2.7). Hence, the proposition is proved and Lemma 2.1 is also proved. □

2.6 Remark The solution of Equation (2.4) with the initial condition $f|_{t=0}\equiv 1$ admits the following formula:

$$f(x,t)=(Id-U\circ\nu)^{-1}(1)=1+U(\nu)+(U\circ\nu\circ U)(\dot{\nu})+\dots \tag{2.8}$$

Indeed, its right-hand side is a well defined C^∞ family of C^∞ functions on $\mathbb{T}^2$, which follows from the uniform boundedness of the operators $(Id-U\circ\nu)^{-1}$ in any given Hilbert Sobolev space. By definition, it satisfies the unit initial condition. Differentiating (2.8) in t yields

$$(Id-U\circ\nu)^{-1}\circ(U\circ\dot{\nu})\circ(Id-U\circ\nu)^{-1}(1)=(Id-U\circ\nu)^{-1}\circ(U\circ\dot{\nu})f(x,t).$$

Hence, the function (2.8) satisfies (2.4).

2.3 Zero of holomorphic differentials. Proof of Proposition 2.2

Let us prove the existence of local holomorphic coordinate. Without loss of generality we assume that $\mu(0)=0$. One can achieve this by applying a real-linear transformation of the plane $\mathbb{R}^2=\mathbb{C}\supset D$ that brings the ellipse at 0 (associated to σ_μ) to a circle. One can achieve also that μ is arbitrarily small with derivatives of orders up to 3 applying a homothety and taking the restriction to a smaller disc centered at 0. We consider that the disc where μ is defined is embedded into $\mathbb{T}^2$ and extend the function μ smoothly to $\mathbb{T}^2$. We assume that the extended function satisfies the inequality $||\mu||_{C^3(\mathbb{T}^2)}<\delta$, and one can make δ arbitrarily small.

Let $\nu(x,t)=t\mu$, $f(x,t)$ be the corresponding function family from Lemma 2.1 constructed as the solution of differential Equation (2.4) with unit initial condition. Put $f(x)=f(x,1)$. We show in the next paragraph that $f(0)\neq 0$ if the constant δ is small enough. Then the local coordinate we are looking for is the function

$$w(z)=\int_0^z f(dz+\mu d\bar{z}).$$

Indeed, it is well-defined and holomorphic by definition. Its local univalence follows from the nondegeneracy of its differential $f(0)(dz+\mu(0)d\bar{z})$ at 0, which, in turn, follows from the inequalities $|\mu|<1$, $f(0)\neq 0$.

Recall that by (2.8),

$$f(x)=(Id-U\circ\mu)^{-1}(1), \text{ where } U=(\partial_{\bar{z}})^{-1}\partial_z.$$

If $\mu=0$, the function $f(x)$ equals 1. Let us show that is is C^0- close to 1 (and hence, $f(0)=f(0,1)\neq 0$), whenever μ is small enough with derivatives up to order 3. Consider the operator-valued functional $\mathcal{A}(\mu)=(Id-U\circ\mu)^{-1}$: its value being an operator acting on the Sobolev space $H^3(\mathbb{T}^2)$, which is well-defined by Proposition 2.5. As it will be shown in the next paragraph, $\mathcal{A}(\mu)$ depends continuously on small functional parameter $\mu\in C^3(\mathbb{T}^2)$,

$max|\mu| < 1$, in the $H^3(\mathbb{T}^2)$ operator norm, and moreover, it has a bounded derivative in μ. Therefore, if $||\mu||_{C^3}$ is small enough, then the function $f(x)$ is close to 1 in H^3 (thus, in C^0, by the Sobolev embedding theorem).

Now for the proof of Proposition 2.2 it is sufficient to prove the boundedness of the derivative $\mathcal{A}'(\mu)$. For any $0 < \delta' < 1$ the operator $\mathcal{A}(\mu)$ is uniformly bounded in all μ with $||\mu||_{C^3} < \delta'$ (Proposition 2.5), so we can apply the usual formula for the derivative of the inverse operator: the derivative of $\mathcal{A}(\mu)$ along a vector $h \in C^3(\mathbb{T}^2)$ is equal to

$$\nabla_h \mathcal{A}(\mu) = \mathcal{A}(\mu) \circ U \circ h \circ \mathcal{A}(\mu). \tag{2.9}$$

To prove the boundedness of the derivative, we have to show that the H^3- norm of the operator in the right-hand side of (2.9) is no greater than some constant (depending on μ) times $||h||_{C^3}$. Indeed, this H^3 operator norm is no greater than $||\mathcal{A}(\mu)||^2_{H^3}$ times the H^3-norm of the operator of multiplication by the function h, the latter is no greater than $||h||_{C^3}$ times some universal constant. This proves the boundedness of the derivative. Proposition 2.2 is proved, also completing the proof of Theorem 1.9.

3 Quasiconformal mappings. Proof of Theorem 1.5

3.1 The plan of the proof of Theorem 1.5

We have already proved the statement of Theorem 1.5 for a C^∞ double-periodic almost complex structure on $\mathbb{C}$: a lifting to the universal cover $\mathbb{C}$ of a C^∞ complex structure on $\mathbb{T}^2$. In this case, the diffeomorphism $\mathbb{C} \to \mathbb{C}$ for the theorem is the lifting to the universal covers of the diffeomorphism of the tori given by Theorem 1.9. To prove Theorem 1.5 in the general case, let σ be a given (possibly just measurable) bounded complex structure on $\mathbb{C}$, we consider a sequence σ_n of C^∞ double-periodic complex structures on $\mathbb{C}$ with growing periods and uniformly bounded dilatations (say less than a fixed $K > 0$) that converge to σ almost everywhere. For each σ_n there is a normalized quasiconformal diffeomorphism $\Phi_n : \mathbb{C} \to \mathbb{C}$ transforming σ_n to the standard complex structure. We show that the diffeomorphisms Φ_n converge (uniformly on $\overline{\mathbb{C}}$) to a homeomorphism Φ. We will prove that Φ is a quasiconformal homeomorphism sending σ to the standard complex structure (see the end of the subsection). The uniqueness of a latter homeomorphism and its diffeomorphicity on a domain where σ is C^∞ will be proved in 3.4. Its analytic dependence on parameter (Theorem 1.6) will be proved in 3.5.

We prove the convergence of Φ_n by equicontinuity of the normalized K-homeomorphisms:

3.1 Lemma *[Ah2]. For any $K > 0$ the normalized K-homeomorphisms $\mathbb{C} \to \mathbb{C}$ (see Definition 1.1) and their inverses are equicontinuous as mappings of the Riemann sphere.*

Lemma 3.1 (proved in 3.2) together with Arzela-Ascoli Theorem imply the following

3.2 Corollary *For any $K > 0$, each sequence of normalized K-homeomorphisms $\mathbb{C} \to \mathbb{C}$ contains a subsequence converging to a homeomorphism $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ uniformly on $\overline{\mathbb{C}}$.*

3.3 Lemma *[Ah2]. Let $K > 0$, $U \subset \mathbb{C}$ be a domain (that may be all of $\mathbb{C}$) $\Phi_n : U \to \Phi_n(U) \subset \mathbb{C}$ be a sequence of K-homeomorphisms converging uniformly on compact subsets to* *a homeomorphism Φ. Let σ_n be almost complex structures sent to the standard one by Φ_n. Let σ_n converge almost everywhere to some σ, and σ have dilatation at most K. Then Φ is a K-homeomorphism sending σ to the standard complex structure.*

Lemma 3.3 will be proved in Subsection 3.3 (using Lemma 3.1 and Corollary 3.2).

Proof of existence in Theorem 1.5 modulo Lemmas 3.1 and 3.3. Let σ_n, σ, K, Φ_n be as at the beginning of the section. Then Φ_n are K-diffeomorphisms. Passing to a subsequence, one can achieve that Φ_n converge to a homeomorphism Φ (Corollary 3.2). By Lemma 3.3, Φ is a K-homeomorphism transforming σ to the standard complex structure. The existence in Theorem 1.5 is proved. □

3.4 Remark In the proof of the existence in Theorem 1.5 we had used only the statements of Lemmas 3.1 and 3.3 for C^∞ diffeomorphisms. Their statements for general quasiconformal homeomorphisms will be used in Subsection 3.4 for the proof of the uniqueness in Theorem 1.5.

3.2 Normality. Proof of Lemma 3.1

The proof of Lemma 3.1 is based on the Grötzsch inequality (the next lemma) comparing moduli of K-homeomorphic complex annuli. To state it, let us first recall the following

3.5 Definition see [Ah2]. The *modulus* of an annulus $A = \{r < |z| < 1\}$ is $m(A) = -\frac{1}{2\pi}\ln r$.

3.6 Remark Consider the cylinder $\mathbb{R} \times S^1$ with the coordinates (x, ϕ), $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and the standard complex structure, which is induced by the Euclidean metric $dx^2 + d\phi^2$.

$$\text{For any } R > 0 \text{ put } A(R) = \{0 < x < R\}; \text{ then } m(A(R)) = \frac{R}{2\pi}. \tag{3.1}$$

The modulus of an annulus is invariant under conformal mappings [Ah2].

3.7 Lemma *(Grötzsch, see [Ah2]). Let $K > 0$, $f : A_1 \to A_2$ be a K-homeomorphism of complex annuli. Then*

$$K^{-1}m(A_1) \leq m(A_2) \leq Km(A_1). \tag{3.2}$$

Proof For the completeness of presentation, we give the classical proof of the lemma. Let us prove the left inequality. First we do this for a K-diffeomorphism; the general case will be treated analogously afterwards. Let us consider that the annuli are drawn on the previous cylinder, say, $A_1 = A(R_1)$, $A_2 = A(R_2)$, then $m(A_i) = \frac{R_i}{2\pi}$, $i = 1, 2$, see (3.1). Thus, it suffices to show that $R_2 \geq K^{-1}R_1$. To do this, consider the pullback (denoted by g) to A_1 under f of the Euclidean metric of A_2 (denote by $| \ |_g$ ($Area_g$) the corresponding norm of vector fields on A_1 (respectively, the area), $Area$ being the Euclidean area). One has $Area(A_i) = 2\pi R_i$, $Area(A_2) = Area_g(A_1)$. We show that

$$Area_g(A_1) \geq K^{-1}Area(A_1). \tag{3.3}$$

This together with the previous formuli for the areas will prove the lemma. For the proof of (3.3) we consider the family $A(r) = \{0 < x < r\} \subset A_1$ of subannuli in A_1, $r \leq R_1$, and prove a lower bound of the derivative $(Area_g(A(r)))'_r$. To do this, consider the vector

Fig.1

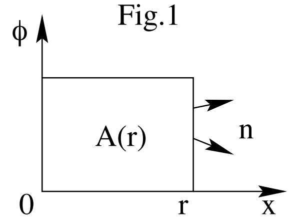

Figure 1: The annuli family $A(r)$ and the vector field n.

field $\frac{\partial}{\partial x}$ as the sum of its component tangent to the circles $x = const$ and the g- orthogonal component (denote the latter component normal to the circles by n, see Fig.1). The vector field n has the same projection to the x- axis, as $\frac{\partial}{\partial x}$ and its flow leaves invariant the fibration by circles $x = const$: its time t flow map transforms the circle $\{x = r\} \subset \partial A(r)$ to the circle $\{x = r + t\} \subset \partial A(r + t)$. Therefore,

$$(Area_g(A(r)))'_r = \int_{x=r,\phi\in[0,2\pi]} \left|\frac{\partial}{\partial\phi}\right|_g |n|_g \, d\phi. \quad (3.4)$$

One has $|n|_g \geq K^{-1}|\frac{\partial}{\partial\phi}|_g$. Indeed, the g- norm $|\ |_g$ of a vector tangent to A_1 is equal to the standard Euclidean norm $|\ |$ of its image under f: $|n|_g = |f_*n|$, $|\frac{\partial}{\partial\phi}|_g = |f_*\frac{\partial}{\partial\phi}|$. By definition, $|\frac{\partial}{\partial\phi}| = 1$, $|n| \geq |\frac{\partial}{\partial x}| = 1 = |\frac{\partial}{\partial\phi}|$. Therefore, by the K- quasiconformality of f (see, Definition 1.1), $|n|_g = |f_*n| \geq K^{-1}|f_*\frac{\partial}{\partial\phi}| = K^{-1}|\frac{\partial}{\partial\phi}|_g$. Hence, the previous derivative is no less than

$$K^{-1}\int_{x=r,\phi\in[0,2\pi]} \left|\frac{\partial}{\partial\phi}\right|_g^2 d\phi \geq K^{-1}(2\pi)^{-1}(\int_{x=r,\phi\in[0,2\pi]} \left|\frac{\partial}{\partial\phi}\right|_g d\phi)^2,$$

by the Cauchy-Bouniakovskii-Schwarz inequality. The latter integral is no less than 2π. Indeed, it is equal to the length in the metric g of the circle $x = r$, or in other terms, the Euclidean length of its image under f, which is a closed curve in A_2 isotopic to a circle $x = const$. Therefore, $(Area_g(A(r)))'_r \geq 2\pi K^{-1}$, thus, $Area_g(A_1) \geq 2\pi K^{-1}R_1 = K^{-1}Area(A_1)$. This proves (3.3) and the left inequality in (3.2) for a K-diffeomorphism f. In the case, when f is a K-homeomorphism, thus just having local L_2 derivatives, the previous discussion remains valid. In more detail, the "area Jacobian integral formula" (the area of the image equals the integral of the Jacobian over the area of the preimage) remains valid for continuous epimorphic mappings of topological degree 1 with L_2 derivatives of (open or closed) planar domains (in particular, for quasiconformal mappings). This follows from the same statement for smooth mappings, density of the latters in the space of continuous mappings with L_2 derivatives (in the topology defined below) and the continuity of area and the Jacobian integral in this topology. Namely, we take the weakest topology for which the Sobolev and C^0 norms are continuous: a mapping sequence converges if and only if it converges uniformly on compact sets and their derivatives converge in L_2. The area of A_2 equals the integral from 0 to R_1 of the right-hand side in (3.4) (the Jacobian formula). This together with the discussion following (3.4) proves the left inequality in (3.2).

Let us prove the right inequality in (3.2). To do this, consider the rectangles

$$R(\psi) = \{0 \leq \phi < \psi\} \subset A_1 = R(2\pi).$$

The vector field $\frac{\partial}{\partial\phi}$ is a sum of its component tangent to the lines $\phi = const$ and the g-orthogonal component (which will be now denoted by n). Repeating the previous discussion, replacing $\frac{\partial}{\partial x}$ by $\frac{\partial}{\partial\phi}$ yields

$$(Area_g(R(\psi)))' \geq K^{-1}R_1^{-1}(\int_{\phi=\psi, 0\leq x\leq R_1} \left|\frac{\partial}{\partial x}\right|_g dx)^2.$$

The latter integral is no less than R_2, since it equals the length of a curve connecting the boundary circles of A_2 (this curve is the image of the subintegral segment). Therefore,

$$Area_g(A_1) = Area(A_2) = 2\pi R_2 \geq K^{-1}2\pi R_1^{-1}R_2^2, \text{ thus, } R_2 \leq KR_1.$$

This together with (3.1) proves the right inequality in (3.2). Lemma 3.7 is proved. □

To prove Lemma 3.1, we need to show that close points cannot be mapped to distant points under a normalized K- homeomorphism or its inverse. This is proved by comparing moduli of appropriate annuli with those of their images (using Lemma 3.7).

For the proof of Lemma 3.1 we recall the notion of the Poincaré metric [CG]. The Poincaré metric of the unit disc $|z| < 1$ is $\frac{4|dz|^2}{(1-|z|^2)^2}$ and is invariant under its conformal automorphisms. A Riemann surface is *hyperbolic*, if its universal covering is conformally equivalent to the unit disc (see Theorem 1.10). For example, any domain in $\mathbb{C}$ whose complement contains more than one point is hyperbolic. The Poincaré metric of a hyperbolic Riemann surface is the pushforward of the Poincaré metric of the unit disc under the universal covering.

3.8 Remark (see [CG]). The Poincaré metric is well-defined, complete and decreasing: the Poincaré metric of a subdomain of a hyperbolic Riemann surface is greater than that of the ambient surface. The Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$ is greater than its standard spherical metric times a constant.

In the proof of Lemma 3.1 we use the following relation of modulus of an annulus and its Poincaré metric, whose proof is a straightforward calculation.

3.9 Proposition *(see, e.g., [DH]). The modulus of an annulus is equal to π times the inverse of the Poincaré length of its closed geodesic.*

Let us prove the equicontinuity of the normalized K-homeomorphisms by contradiction. Suppose the contrary: there exist an $\varepsilon > 0$, a sequence of normalized K-homeomorphisms $\Phi_n : \mathbb{C} \to \mathbb{C}$ and a sequence of pairs $x_n, y_n \in \mathbb{C}$, $|x_n - y_n| \to 0$, $|\Phi_n(x_n) - \Phi_n(y_n)| > \varepsilon$ (in the spherical metric of $\overline{\mathbb{C}}$). Without loss of generality we assume that the sequence x_n (and hence, y_n) converges (one can achieve this by passing to a subsequence, denote by x the limit). Then there is a sequence A_n of annuli in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ bounded by circles centered at x and surrounding the pairs x_n, y_n: one of the circles is fixed, the other one contracts to x, as $n \to \infty$ see Fig.2. By definition, the annuli A_n tend to once punctured disc, hence, $m(A_n) \to \infty$. The point x may coincide with one of the three points 0, 1, ∞. Let us take two of the latter that are distinct from x (say, let them be 0, 1). Then each annulus A_n separates the pairs (x_n, y_n) and $(0, 1)$. By Lemma 3.7, $m(\Phi_n(A_n)) \to \infty$ as well (the left inequality in (3.2)). Hence, by Proposition 3.9, the lengths of the geodesics (denoted by γ_n) of the annuli $\Phi_n(A_n)$ in their Poincaré metrics tend to zero. But the latter lengths are greater than the lengths of γ_n taken in the Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$, and hence, also greater than their lengths in the spherical metric times a constant independent from n (by Remark 3.8). Thus, each γ_n separates the pairs $(\Phi_n(x_n), \Phi_n(y_n))$ and $(0, 1)$ and is a closed curve with spherical length tending to 0. Hence, the spherical distance between $\Phi_n(x_n)$ and $\Phi_n(y_n)$ tends to 0 - a contradiction. The equicontinuity of the inverses to the normalized K-homeomorphisms is proved analogously by using the right inequality in (3.2). Lemma 3.1 is proved.

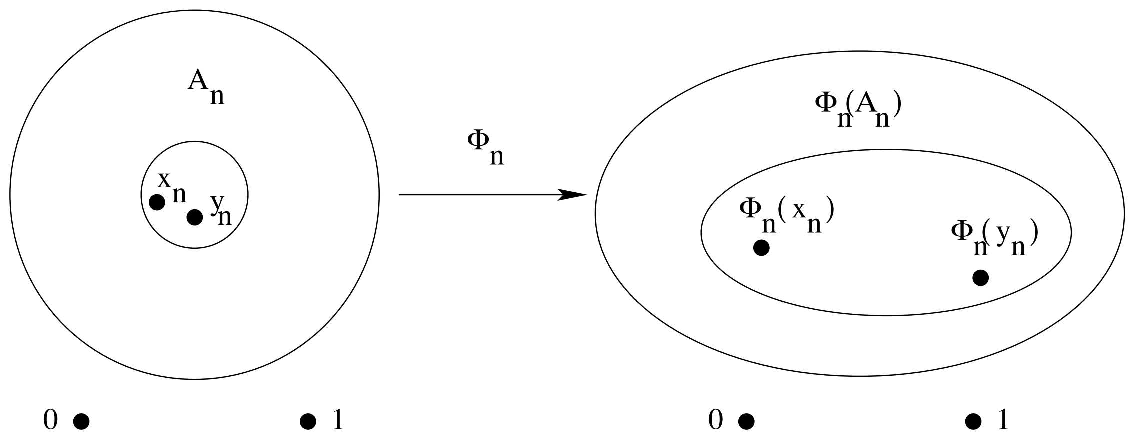

Figure 2: A homeomorphism sending close points to distant ones distorts an annulus.

3.3 Quasiconformality and weak convergence. Proof of Lemma 3.3

Let Φ_n, σ_n, Φ, σ be as in Lemma 3.3. Let us show that Φ is quasiconformal, more precisely: 1) Φ has local L_2 derivatives that are weak L_2 limits of those of Φ_n; 2) Φ transforms σ to the standard complex structure (and hence, is K- quasiconformal, see Remark 1.17). This will prove Lemma 3.3.

In the proof of statement 1) we use the following

3.10 Proposition *Let Φ_n be a sequence of continuous mappings of a domain $U \subset \mathbb{C}$ to $\mathbb{C}$. Let $\Phi_n \to \Phi$ uniformly on compact sets in U. Let in addition, Φ_n have local L_2 distributional derivatives, and for any $D \Subset U$ the norms $||d\Phi_n||_{L_2(D)}$ be uniformly bounded. Then the limit Φ also has local L_2 distributional derivatives that are the L_2- weak limits of those of Φ_n.*

Proof Given an arbitrary $D \Subset U$, passing to a subsequence one can achieve that the derivatives of the mappings Φ_n converge L_2- weakly on D (boundedness). On the other hand, they converge to the derivative of Φ in sense of distributions. Therefore, the latter is also L_2 locally and the whole sequence of the derivatives converges L_2- weakly. (For a bounded sequence in $L_2(D)$, weak convergence in L_2 is equivalent to the convergence in the sense of distributions. □

3.11 Corollary *For any $K > 0$ the statements of Proposition 3.10 hold true for any sequence of K-homeomorphisms $\Phi_n : U \to \Phi_n(U) \subset \mathbb{C}$ converging uniformly on compact subsets.*

Proof On each domain $D \Subset U$ one has $||d\Phi_n||^2_{L_2(D)} \leq KArea(\Phi_n(D))$, which follows from the definition, K- quasiconformality and the Jacobian integral formula for the area of image (valid for quasiconformal mappings, see the proof of Lemma 3.7). The latter areas (and hence, the previous L_2- norms) are uniformly bounded, since Φ_n converge uniformly on $\overline{D}$. This together with Proposition 3.10 proves the corollary. □

The corollary implies statement 1). Let us prove statement 2). Let μ_n, μ be the functions from (1.1) defining the complex structures σ_n and σ respectively, thus, $d\Phi_n = f_n(dz + \mu_n d\bar{z})$. By assumption, $|\mu_n| < 1$, $\mu_n \to \mu$ almost everywhere. We have to show that the function Φ satisfies Beltrami Equation (1.2) almost everywhere (Remark 1.17). Indeed, $f_n \to f = \frac{\partial \Phi}{\partial z}$, $f_n\mu_n \to \frac{\partial \Phi}{\partial \bar{z}}$ (both L_2 weakly), as $n \to \infty$; f_n are uniformly bounded in each local space $L_2(D)$. The functions μ_n are uniformly bounded by 1 and converge almost everywhere. Therefore, the weak limit of their products $f_n\mu_n$ is the product $f\mu$ of their limits. This proves that Φ satisfies Equation (1.2), together with statement 2) and Lemma 3.3.

3.4 Uniqueness, smoothness and group property

The existence of the normalized quasiconformal homeomorphism from Theorem 1.5 is proved (see 3.1). Here we prove its uniqueness and the group property of quasiconformal mappings (Proposition 1.3). The uniqueness follows from the local uniqueness up to composition with a conformal mapping and from the normalization. The local uniqueness (together with the diffeomorphicity on a domain where the almost complex structure is C^∞) are implied by the following:

3.12 Proposition *Let $D \subset \mathbb{C}$ be a simply-connected domain, σ be a bounded measurable almost complex structure on D and $\Phi : D \to \Phi(D) \subset \mathbb{C}$ be a quasiconformal homeomorphism transforming σ to the standard complex structure. Then Φ is unique up to left composition with a conformal mapping. It is a C^∞ diffeomorphism, if σ is C^∞.*

Proof Let $\mu : D \to \mathbb{C}$ be the function defining the almost complex structure σ.

Case $\mu \equiv 0$**.** In this case Proposition 3.12 is implied by the following

3.13 Proposition *Let $D \subset \mathbb{C}$ be a simply connected domain. Let $\Phi : D \to \mathbb{C}$ be a continuous mapping having local L_2 distributional derivatives and $\frac{\partial \Phi}{\partial \bar{z}} \equiv 0$ almost everywhere. Then Φ is holomorphic.*

Proof Fix a $z_0 \in D$ and put $U(z) = \int_{z_0}^{z} \Phi(\zeta) d\zeta$. We show that the function $U(z)$ is well-defined (independent on the choice of path connecting z_0 to z). Then it is holomorphic by definition, hence, so is $\Phi(z) = \frac{\partial U}{\partial z}$. It suffices to show that the integral of the form Φdz along any Jordan curve is zero. Since the derivatives of Φ are locally L_2, we can apply the Stokes formula: the previous integral is equal to the integral of the differential $d(\Phi dz)$ over the domain bounded by the curve. But $d(\Phi dz) = \frac{\partial \Phi}{\partial \bar{z}} d\bar{z} \wedge dz = 0$, so, the integral is zero. □

Case $\mu \in C^\infty$**.** Each point of D has a neighborhood where there exists at least one C^∞ quasiconformal diffeomorphism Ψ transforming σ to the standard complex structure (Proposition 2.2). The composition $\Phi \circ \Psi^{-1}$ preserves the standard complex structure by definition and is quasiconformal: it has local L_2 derivatives, since so does Φ and since Ψ^{-1} is C^∞. Therefore, it is conformal, as is proved above, hence, Φ is a C^∞ diffeomorphism. This already proves the smoothness of the K-homeomorphism from Theorem 1.5 in the domains where the almost complex structure is C^∞.

Case μ is measurable. In the proof of the uniqueness in this case we use the following

3.14 Proposition *Any homeomorphism of domains in $\mathbb{C}$ with local L_2 distributional derivatives transforms any set of measure zero to a set of measure zero.*

Proof For any bounded measurable subset V in the definition domain of the homeomorphism the area of the image of V is no greater than the L_2- norm of its differential on V. Therefore, the image has measure zero, if so does V. □

Let $0<\delta<1$, $|\mu|<\delta$, $K=\frac{1+\delta}{1-\delta}$, μ_n be a sequence of C^∞ functions, $|\mu_n|<\delta$, $\mu_n\to\mu$ almost everywhere. (We extend μ_n, μ to $\mathbb{C}$ with the latter inequality and convergence so that $\mu_n\in C^\infty(\mathbb{C})$.) Consider the corresponding almost complex structures σ_n and σ on $\mathbb{C}$, which have dilatations at most K by construction. Let $\Psi_n:\mathbb{C}\to\mathbb{C}$ be K-diffeomorphisms transforming σ_n to the standard complex structure (the latter exist as is proved above). Passing to subsequence, one can assume that they converge uniformly on $\overline{\mathbb{C}}$ (by Corollary 3.2). Denote by Ψ their limit, which is a K-homeomorphism $\mathbb{C}\to\mathbb{C}$ transforming the extended complex structure σ to the standard one (Lemma 3.3). It suffices to show that $h=\Phi\circ\Psi^{-1}:\Psi(D)\to\Phi(D)$ is a conformal homeomorphism. It is a homeomorphism, since so are Φ and Ψ. To prove its conformality, we show that h satisfies the conditions of Proposition 3.13. To do this, consider the homeomorphisms $h_n=\Phi\circ\Psi_n^{-1}:\Psi_n(D)\to\Phi(D)$. They are quasiconformal homeomorphisms with dilatations at most K^2, as in the previous case (see Remark 1.2). They converge to h uniformly on compact subsets of $\Psi(D)$. Hence, h has local L_2 distributional derivatives (Corollary 3.11).

Now let us show that $\frac{\partial h}{\partial\bar z}=0$ almost everwhere. The homeomorphisms $\Psi^{\pm1}$ have local L_2 distributional derivatives, since they are limits of $\Psi_n^{\pm1}$, which are K-diffeomorphisms (Corollary 3.11 and Remark 1.2). For any homeomorphism $\Psi^{\pm1}$ the set of its nondifferentiability points and the zero locus of its derivative have measure zero. Indeed, the zero locus of the derivative of $\Psi^{\pm1}$ is contained in the image under $\Psi^{\mp1}$ of the nondifferentiability locus of the latter; the nondifferentiability locus (and hence, its image) have measure zero by the L_2-differentiability and Proposition 3.14. For almost any point $z\in\mathbb{C}$ the derivative $d\Psi^{-1}(z)$ exists and is nondegenerate and transforms the standard complex structure to σ; the derivative $d\Phi(\Psi^{-1}(z))$ exists and either is zero, or transforms σ to the standard complex structure. This follows from the previous statement and definition. Therefore, $\frac{\partial h}{\partial\bar z}=0$ almost everywhere. This together with the local L_2- differentiability of h, Proposition 3.13 and Remark 1.17 prove its conformality. Proposition 3.12 is proved and hence, Theorem 1.5 is proved. □

3.15 Proposition *Any quasiconformal homeomorphism and its inverse transform a set of measure zero to a set of measure zero.*

Proof Let $\Phi:U\to\Phi(U)\subset\mathbb{C}$ be a quasiconformal homeomorphism of open subsets in $\mathbb{C}$. Let σ be a bounded almost complex structure on U sent by Φ to the standard complex structure. Let us extend σ up to a measurable bounded almost complex structure on $\mathbb{C}$, let K be the dilatation of the extended structure. Let σ_n be a sequence of C^∞ almost complex structures on $\mathbb{C}$ with dilatations at most K that converge to σ almost everywhere. Let Φ_n be the corresponding normalized quasiconformal homeomorphisms from Theorem 1.5. Passing to a subsequence, one can achieve that they converge to a quasiconformal homeomorphism $\widetilde{\Phi}:\mathbb{C}\to\mathbb{C}$ transforming σ to the standard complex structure (Lemma 3.3). The mappings $\widetilde{\Phi}^{\pm1}$ are limits of K- diffeomorphisms $\Phi_n^{\pm1}$, and hence, have local L_2 derivatives (Corollary 3.11). Therefore, any of them sends a set of measure zero to a set of measure zero (Proposition 3.14). The same is true for the mapping $\Phi^{\pm1}$, which is a composition of $\widetilde{\Phi}^{\pm1}$ with a conformal mapping (Theorem 1.5). □

Proof of Proposition 1.3. The statement of the proposition is local: it suffices to show that compositions (and inverses) of local K-homeomorphisms are K^2- (respectively, K-) quasiconformal. We prove this statement for composition (for inverses the proof is analogous): given domains $U,V,W\subset\mathbb{C}$ and K-homeomorphisms $\Psi:U\to V$, $\Phi:V\to W$, let us show that $h=\Phi\circ\Psi$ is a K^2-homeomorphism. Let σ be an almost complex structure on V sent to the standard one by Φ with dilatation at most K. Let C^∞ almost complex structures σ_n on $\mathbb{C}$, the K-diffeomorphisms $\Phi_n:\mathbb{C}\to\mathbb{C}$ and the K-homeomorphism $\widetilde{\Phi}$ be as in the previous proof. The mappings $\Phi_n\circ\Psi$ are K^2-homeomorphisms (as in the proof of Proposition 3.12, C^∞ case). Their limit $\widetilde{h}=\widetilde{\Phi}\circ\Psi$ has local L_2 distributional derivatives (Corollary 3.11). Let σ' be an almost complex structure on U sent to σ by Ψ (with dilatation at most K^2). For almost any point $z\in U$ the differential $d\Psi(z)$ is well-defined, nondegenerate and sends σ' to σ; the differential $d\widetilde{\Phi}(\Psi(z))$ is well-defined, nondegenerate and sends σ to the standard complex structure. This follows from definition and Proposition 3.15, as at the end of the proof of Proposition 3.12. Therefore, the composition $\widetilde{h}=\widetilde{\Phi}\circ\Psi$ is locally L_2 differentiable and sends σ' to the standard complex structure. Hence, it is a K^2-homeomorphism, and so is h, which is the composition of $\widetilde{h}$ with a conformal mapping (Proposition 3.12 applied to Φ). Proposition 1.3 is proved. □

3.5 Analytic dependence on parameter. Proof of Theorem 1.6

Double-periodic smooth case. Consider a family of double-periodic C^∞ almost complex structures $\sigma(t)$ on $\mathbb{C}$ depending holomorphically on a complex parameter t (this means that the corresponding function $\mu=\mu(z,t)$ from (1.1) is holomorphic in t). We assume that the periods are fixed, thus, $\sigma(t)$ is the lifting to the universal cover $\mathbb{C}$ of an analytic family of almost complex structures on the two-torus. Then the corresponding quasiconformal diffeomorphisms (denoted by Φ_t) from Theorem 1.5 are holomorphic in t as well. Indeed, their differentials are uniformizing differentials. Hence, for any t, $d\Phi_t=f_t(dz+\mu(z,t)d\bar z)$ up to multiplication by complex constant depending on t, where f_t is given by Formula (2.8) with $\nu=\mu(z,t)$. The right-hand side of (2.8) is analytic in the functional parameter ν, hence, f_t is holomorphic in t, as is μ, and $z\mapsto\int_0^z f_t(dz+\mu(z,t)d\bar z)$ is a holomorphic family of diffeomorphisms of $\mathbb{C}$. The family Φ_t is obtained from the latter one by multiplication by a function in t that makes the previous diffeomorphisms normalized (fixing 1), hence, the multiplier function (and thus, Φ_t as well) are also holomorphic in t. Theorem 1.6 is proved in the double-periodic smooth case.

General case. Now consider arbitrary analytic family $\sigma(t)$ of bounded almost complex structures on $\mathbb{C}$ depending on a complex parameter t. (We suppose that t runs through the unit disc D.) Let $\mu(z,t)$ be the corresponding functions, see (1.1), which are holomorphic in t. There exists a $0<\delta<1$ such that $|\mu(z,0)|<\delta$ for any z. The corresponding mapping $M_z:t\mapsto\mu(z,t)$ is a holomorphic mapping $D\to D$ depending on z in a measurable way such that $|M_z(0)| < \delta$. (Recall that for a given $\delta < 1$ the space of holomorphic mappings $M : D \to D$ with $|M(0)| \leq \delta$ is compact, see [CG].) Conversely, for any $0 < \delta < 1$ each measurable collection of holomorphic mappings $M_z : D \to D$ with $|M_z(0)| < \delta$ defines an analytic family of bounded almost complex structures; they are uniformly bounded when restricted to a smaller parameter disc $D_r = \{|t| < r\}$, $r < 1$. Indeed, in the case that $M_z(0) = 0$, one has $|M_z|_{D_r} < r$ (Schwarz Lemma); the general case is easily reduced to the previous one.

Denote by Φ_t the corresponding normalized quasiconformal homeomorphisms from Theorem 1.5. To prove the analyticity of Φ_t in t, we approximate $\sigma(t)$ (in the topology of convergence almost everywhere) by analytic families $\sigma_n(t)$ of C^∞ double-periodic almost complex structures depending holomorphically on the same parameter t, with growing periods $2n$, $2in$, $\sigma_n \to \sigma$. For example, consider the restriction of σ to the period square centered at 0 and take $\widetilde{\sigma}_n$ to be its double-periodic extension. The corresponding previous holomorphic function family $M_z(t) = \widetilde{M}_{z,n}(t) : D \to D$ is measurable in z. Then we approximate it (in the previous topology) by an analytic function family $M_{z,n} : D \to D$ that is C^∞ in z and is periodic with the previous periods. (The approximation should be sharper, when n is larger.) The new function family $M_{z,n}(t)$ defines an almost complex structure family $\sigma_n(t)$ we are looking for. The almost complex structures $\sigma_n(t)|_{t\in D_r}$ are uniformly bounded by construction and the previous discussion.

Denote by $\Phi_{n,t}$ the normalized quasiconformal homeomorphisms transforming $\sigma_n(t)$ to the standard complex structure. They depend analytically on t, as is proved above. For any t, z, $\Phi_{n,t}(z) \to \Phi_t(z)$, as $n \to \infty$ (Lemma 3.3 and the uniqueness in Theorem 1.5). Thus, for any $z \in \mathbb{C}$ $\Phi_t(z)$ is a function in t that is a limit of pointwise converging sequence of holomorphic functions. Let us prove that for any fixed z the functions $\Phi_{n,t}(z)$ in $t \in D_r$ are bounded uniformly in n, so that their limit $\Phi_t(z)$ is holomorphic. Indeed, the almost complex structures $\sigma_n(t)|_{t\in D_r}$ are uniformly bounded. Therefore, the family $\Phi_{n,t}$ (depending on the two parameters n and $t \in D_r$) together with their inverses is equicontinuous (Lemma 3.1). Hence, the previous functions are uniformly bounded, so their limit is holomorphic. Theorem 1.6 is proved.

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