Quasiconformal mappings, complex dynamics and moduli spaces

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Contents

The classical Poincaré–Koebe Uniformization theorem states that *each simply connected Riemann surface is conformally equivalent to either the Riemann sphere, or complex line* $\mathbb{C}$, *or unit disk* D_1. The quasiconformal mapping theory created by M.A.Lavrentiev and C. Morrey Jr. extends the Uniformization Theorem to nonstandard complex structures that may vary from point to point and what is the most important, in a just measurable, may be discontinuous way. It has many very important applications in many different domains of mathematics, first of all, complex analysis, complex dynamics, Kleinian groups and moduli spaces. In the present course we will introduce the quasiconformal mappings and state and prove the main theorem of the theory: the Measurable Riemann Mapping Theorem proved by M.A.Lavrentiev in C^0-case and C. Morrey Jr. in the general measurable case. We provide introduction into holomorphic dynamics, Kleinian groups and the moduli spaces and discuss the above-mentioned applications of the quasiconformal mapping theory, including proofs of Sullivan No Wandering Domain Theorem in rational dynamics and Ahlfors Finiteness Theorem in the Kleinian groups.

1 Almost complex structures, integrability, quasiconformal mappings

1.1 Almost complex structures and quasiconformal mappings. Main theorems

A linear complex structure on $\mathbb{R}^2$ is a structure of a linear space over $\mathbb{C}$. We fix an orientation on $\mathbb{R}^2$ and consider it to be compatible with the complex structure. An *(almost) complex structure* on a real two-dimensional surface is a family of linear complex structures on the tangent planes at its points. A linear complex structure on $\mathbb{R}^2$ defines an ellipse in $\mathbb{R}^2$ centered at 0, which is an orbit under the S^1- action by multiplication by complex numbers with unit modulus. (This ellipse is unique up to a homothety. The ellipse corresponding to the standard complex structure on $\mathbb{C}$ is a circle.) The *dilatation* of a nonstandard linear complex structure on $\mathbb{C}$ (with respect to the standard complex structure) is the *aspect ratio* of the corresponding ellipse. This is the ratio of the largest radius over the smallest one. An almost complex structure defines an ellipse field in the tangent planes, and vice versa: an ellipse field determines the almost complex structure in the unique way.

If our surface is a Riemann surface (with a fixed complex structure), then any (nonstandard) almost complex structure has a well-defined dilatation at each point of the surface. The *(total) dilatation* of an almost complex structure is the essential supremum of its dilatations at all the points. This is the minimal supremum of the dilatations after possible correction of the almost complex structure over a measure zero set. An almost complex structure is said to be *bounded*, if its total dilatation is finite.

Each real linear isomorphism $\mathbb{C} \to \mathbb{C}$ acts on the space of the ellipses centered at 0, and hence, on the space of linear complex structures. Its *dilatation* is defined to be the dilatation of the image of the standard complex structure. It is equal to the aspect ratio of the image of a circle centered at 0. The action of a differentiable homeomorphism of domains in $\mathbb{C}$ on the almost complex structures and its dilatation (at a point) are defined to be those of its derivative (at the points where the derivative exists and is a nondegenerate linear operator). At those points where the derivative exists and is a nonzero degenerate operator, the dilatation is defined to be infinite. The *(total)* dilatation is the essential supremum of the dilatations through all the previous points.

It was proved at the beginning of XX-ths century [17, 19] that *any C^∞ almost complex structure is locally integrable*, that is, can be transformed to the standard complex structure by a local C^∞ diffeomorphism (a priori, defined in a small neighborhood of a given point).

Remark 1.1 The proof of the local integrability of an analytic almost complex structure is elementary and due to Gauss. It is done immediately by analyzing the complexification of the corresponding $\mathbb{C}$- linear 1- form. But the proof is already nontrivial in the C^∞ case.

The next theorem is a much stronger statement about global integrability of a measurable bounded almost complex structure. To state it, let us recall the following definition.

Definition 1.2 (see [3]). Let $K > 0$. A diffeomorphism of domains in $\mathbb{C}$ is said to be a K*-diffeomorphism*, if its dilatation is no greater than K. A homeomorphism of domains in $\mathbb{C}$ is said to be K- *quasiconformal* (or K-homeomorphism), if it has local L_2 distributional derivatives and its total dilatation is no greater than K. A homeomorphism is said to be quasiconformal if it is K- quasiconformal for some $K > 0$.

Remark 1.3 The dilatations of a two-dimensional linear operator and of its inverse are equal. The inverse to a K-diffeomorphism is also a K-diffeomorphism. The composition of two K-diffeomorphisms is a K^2-diffeomorphism. This follows from the definition.

Proposition 1.4 *[3]. The quasiconformal homeomorphisms of a Riemann surface form a group.*

This proposition will be proved later on.

Definition 1.5 A homeomorphism $\mathbb{C} \to \mathbb{C}$ is said to be *normalized*, if it fixes 0 and 1.

Theorem 1.6 *(Existence: M.A.Lavrentiev (C^0-case, [18]) and C. Morrey, Jr. (general case, [21])). For any measurable bounded almost complex structure σ on $\mathbb{C}$ there exists a unique normalized quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ that transforms σ to the standard complex structure (at almost any point outside the zero locus of its derivative). If σ is C^∞ in some domain, then the homeomorphism is a C^∞ diffeomorphism while restricted to this domain.*

Remark 1.7 In the case, when σ is the standard complex structure, the quasiconformal homeomorphism from the theorem is the identity (uniqueness). The next example shows that the condition that the distributional derivatives of a quasiconformal mapping are locally L_2 is important. Namely, the uniqueness statement of Theorem 1.6 is not true for homeomorphisms that are not necessarily quasiconformal but just differentiable almost everywhere (even with bounded dilatation).

Example 1.8 There exists a homeomorphism $H : \mathbb{C} \to \mathbb{C}$ that is holomorphic almost everywhere but not holomorphic everywhere. Indeed, consider the standard Cantor set $K \subset [0, 1]$ of one-dimensional Lebesgue measure zero that is obtained by deleting the middle third part $(\frac{1}{3}, \frac{2}{3})$, then the middle third parts in the remaining two segments etc. Consider the Cantor staircase function $\phi : [0, 1] \to [0, 1]$: a monotonic function that is constant on the deleted intervals and that sends K onto the unit segment: $\phi(1) = 1$. Let us extend it to the function $\phi : \mathbb{R} \to [0, 1]$ by setting $\phi|_{(-\infty,0]} \equiv 0$, $\phi|_{[1,+\infty)} \equiv 1$. The mapping

$$H(z) := z + i\phi(x).$$

is a homeomorphism $\mathbb{C} \to \mathbb{C}$ with the required properties.

1.2 The Beltrami Equation. Dependence of the quasiconformal homeomorphism on parameter

To a nonstandard almost complex structure, denoted by σ, on a subset $D \subset \mathbb{C}$ we put into correspondence a $\mathbb{C}$- valued 1- form that is $\mathbb{C}$- linear with respect to σ. The latter form can be normalized to be

$$\omega_\mu = dz + \mu(z)d\bar{z}, \ |\mu| < 1. \tag{1.1}$$

The function μ is uniquely defined by σ. Conversely, for an arbitrary complex-valued function μ, with $|\mu| < 1$, the 1- form ω_μ defines the unique complex structure for which ω_μ is $\mathbb{C}$- linear. We denote by σ_μ the almost complex structure thus defined (whenever the contrary is not specified). Then σ_μ is bounded, if and only if the essential supremum of the function $|\mu|$ is less than 1. In what follows we denote

$$\sigma_{st} = \sigma_0 = \text{ the standard complex structure.}$$

Remark 1.9 The ellipse associated to σ_μ on the tangent plane at a point z is given by the equation $|dz + \mu(z)d\bar{z}| = 1$; the dilatation (aspect ratio) is equal to $\frac{1+|\mu(z)|}{1-|\mu(z)|}$. Sometimes an almost complex structure is represented by an invariant object: its *Beltrami differential*

$$\mu(z)\frac{d\bar{z}}{dz}, \ |\mu| < 1.$$

And vice versa: *each "tensor of type"* $\mu(z)\frac{d\bar{z}}{dz}$ *of norm less than one at each point (i.e., with* $|\mu| < 1$*) is the Beltrami differential of some almost complex structure.*

We will be looking for a differentiable homeomorphism $\Phi(z)$ that is holomorphic; it transforms σ_μ to the standard complex structure. This is equivalent to say that the differential of Φ (which is a closed form) is a $\mathbb{C}$- linear form, or equivalently, has the type $f(z)(dz + \mu d\bar{z})$:

$$\frac{\partial \Phi}{\partial \bar{z}} = \mu \frac{\partial \Phi}{\partial z} \text{ (Beltrami equation).} \tag{1.2}$$

Remark 1.10 Theorem 1.6 is equivalent to the statement that the Beltrami Equation (1.2) with any measurable functional coefficient μ, $|\mu| < c < 1$ has a quasiconformal homeomorphic solution $\Phi : \mathbb{C} \to \mathbb{C}$.

Theorem 1.11 *(Dependence on Parameters: Ahlfors and Bers [4]). Consider a family of almost complex structures on* $\mathbb{C}$ *that depends continuously on some parameter and have uniformly bounded dilatations. Then the corresponding normalized quasiconformal homeomorphisms from Theorem 1.6 also depend continuously on the parameter. Let now the family of almost complex structures depend analytically on complex parameter* t*: that is, the corresponding family of functions* $\mu = \mu(z, t)$ *is holomorphic in the parameter. Then the corresponding homeomorphism from Theorem 1.6 also depends analytically on the parameter.*

The classical proofs of Theorems 1.6 and 1.11 are tricky and use delicate theorems of analysis like Calderon–Zygmund inequality. A simplified proof of Theorem 1.6 using Fourier transform was given in [13]. In our course we will present proofs of Theorem 1.6 and 1.11 given in [11] that seem to be simpler than the above-mentioned proofs. They use only elementary Fourier series analysis and Sobolev space arguments.

The measurable versions of Theorems 1.6 and 1.11 have many very important applications in various domains of mathematics, especially in holomorphic dynamics and the theory of Kleinian groups. We will discuss them in the course. More specifically, in holomorphic dynamics a technique called quasiconformal surgery depends upon invariant almost complex structures that are discontinuous.

In order to prove Theorem 1.6, we first prove its version for C^∞ almost complex structures on the two-torus following [11]. The proof uses only the elementary Fourier analysis.

Theorem 1.12 *([1]) For any C^∞ almost complex structure σ on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ (which is always bounded) there exists a C^∞ diffeomorphism of $\mathbb{T}^2$ onto an appropriate complex torus, depending on σ, that transforms σ to the standard complex structure.*

We then deduce Theorem 1.6 from Theorem 1.12 by using double-periodic approximations of a given almost complex structure on $\mathbb{C}$ and simple normality arguments involving a Grötzsch inequality for annuli diffeomorphisms. This deduction follows the classical scheme [3].

For the proof of Theorem 1.12 we prove the existence of a global nowhere vanishing σ-holomorphic differential. To do this, we use the homotopy method for the Beltrami equation with a parameter, which reduces the proof to solving a linear ordinary differential equation in $L_2(\mathbb{T}^2)$. We prove regularity of its solution by showing that the equation is bounded in any Sobolev space $H^s(\mathbb{T}^2)$.

Remark 1.13 The proof of Theorem 1.12 presented below originally appeared in a previous paper [12], where the same method was used to prove a foliated version of Theorem 1.6.

We also give a proof of the classical Poincaré-Köbe uniformization theorem using Theorem 1.12 (modulo a topological statement on simply-connected Riemann surfaces):

Theorem 1.14 *[15, 16, 22]. Each simply-connected Riemann surface is conformally equivalent to either the unit disc, $\mathbb{C}$, or the Riemann sphere.*

2 Complex dynamics

The dynamics of iterations of rational self-mappings of the Riemann sphere was born at the beginning of the XX-th century, when Fatou and Julia have suggested to split the Riemann sphere into two completely invariant subsets for the underlying rational function: the Fatou set (the maximal open subset, where the iterates form a normal family) and its complement, the Juila set. Afterwards very many remarkable results were obtained, together with beautiful fractal pictures for the Julia sets. Introduction of the quasiconformal mapping to the theory in 1980-th led to a major breakthrough: the famous Sullivan No Wandering Domain Theorem, which completed the description of possible dynamics on the Fatou set. Since then, the quasiconformal mapping theory became widely used in complex dynamics and led to many other very important results. We give a survey of dynamics of rational iterations, together with main open problems and proofs of some basic theorems including the above-mentioned Sullivan Theorem.

2.1 Normal families. Montel Theorem

Definition 2.1 A Riemann surface is said to be *hyperbolic (parabolic)*, if its universal covering is conformally equivalent to the unit disk (respectively, the complex line $\mathbb{C}$).

Example 2.2 A domain in $\overline{\mathbb{C}}$ is parabolic if and only if it is once or twice punctured Riemann sphere: every other domain, including triple punctured Riemann sphere is hyperbolic. The universal covering of triple punctured Riemann sphere by unit disk is given by the classical modular function constructed by reflected copies of an ideal triangle in the unit disk.

Definition 2.3 A family of mappings $(f_s)_{s\in S}: U \to V$ of locally compact metric spaces is said to be *normal,* if each their subsequence contains a subsequence converging uniformly on compact subsets. (If the mappings under question are continuous, the normality is equivalent to the equicontinuity condition, by Arzela–Ascoli Theorem.)

Theorem 2.4 *(Montel). Let $a, b, c \in \overline{\mathbb{C}}$ be three distinct points. Let $(f_s)_{s\in S}: U \to \overline{\mathbb{C}}$ be a family of holomorphic mappings of a Riemann surface U to $\overline{\mathbb{C}}$ that avoid the values a, b, c. Then the family f_s is normal.*

Remark 2.5 Montel Theorem is essentially implied by the following facts:

- the standard Lobachevsky–Poincaré metric of the unit disk induces a well-defined metric on each hyperbolic Riemann surface (called the *Poincaré metric*),

- each holomorphic mappings between hyperbolic Riemann surfaces does not increase distances in the Poincaré metric (Schwartz Lemma in invariant form).

See the corresponding exercises in Task 2.

In what follows we will use the following slight generalization of Montel Theorem

Theorem 2.6 *(Holomorphic Variable Montel Theorem). Let $g_1, g_2, g_3 : U \to \overline{\mathbb{C}}$ be three holomorphic mappings with pointwise distinct values: $g_1(z)$, $g_2(z)$, $g_3(z)$ are distinct for every $z \in U$. Let $(f_s)_{s\in S}: U \to \overline{\mathbb{C}}$ be a family of holomorphic mappings such that for every $s \in S$ one has $f_s(z) \neq g_j(z)$ for $j = 1, 2, 3$. Then the family $(f_s)_{s\in S}$ is normal.*

Proof For every $z \in U$ let $h_z \in Aut(\overline{\mathbb{C}})$ denote the conformal automorphism of the Riemann sphere that sends 0, 1 and ∞ to $g_1(z)$, $g_2(z)$ and $g_3(z)$ respectively. It depends holomorphically on the parameter z. For every $s \in S$ set $F_s(z) = h_z^{-1}(f_s(z))$. The family $(F_s)_{s\in S}: U \to \overline{\mathbb{C}}$ is normal, by Montel Theorem and since $F_s(z) \neq 0, 1, \infty$: the latter inequality follows from the assumption that $f_s(z) \neq g_j(z)$ and construction, since h_z^{-1} sends the forbidden value $g_j(z)$ to some of points 0, 1, ∞. Therefore, the family of mappings $f_s(z) = h_z(F_s(z))$ is also normal, by holomorphic dependence of the automorphism family h_z on z. The theorem is proved. □

2.2 Rational dynamics: basic theory

Everywhere below

$$R(z) = \frac{P(z)}{Q(z)}$$

is a rational function considered as a holomorphic self-mapping of the Riemann sphere. Recall that the *degree* of a rational function is its topological degree: the number of preimages of a noncritical value. It is equal to the maximal of the degrees of the polynomials $P(z)$, $Q(z)$ normalized to be coprime.

Definition 2.7 The *Fatou set* $F = F_R \subset \overline{\mathbb{C}}$ of the function R is the set of those points $z \in \overline{\mathbb{C}}$ for which there exists a neighborhood $U = U(z) \subset \overline{\mathbb{C}}$ where the iterates $R^n|_U$ form a normal family. In other words, the Fatou set is the maximal open subset in $\overline{\mathbb{C}}$ where the iterates form a normal family. The *Julia set* is its complement $J = J_R = \overline{\mathbb{C}} \setminus F$.

Example 2.8 The Fatou set of the function $R(z) = z^d$ consists of two components: the interior and the exterior of the unit circle. The iterates converge to 0 and to infinity uniformly on compact subsets in the interior (respectively, exterior) of the unit circle. The Julia set is the unit circle.

Remark 2.9 It follows from definition that

- the Fatou and Julia sets are invariant and completely invariant (that is, invariant under taking the pullback):

$$F = R(F) = R^{-1}(F), \ J = R(J) = R^{-1}(J);$$

- the Fatou and Julia sets of a rational function R coincide with those of any of its fixed iterate R^m;

- the Fatou set is open, and the mapping R sends each its connected component to some connected component of the Fatou set and presents a branched covering of one component over the other.

Let us describe those points that are always contained in the Fatou set. To do this, let us introduce the following definition.

Definition 2.10 Consider a germ $f(z) = \lambda z(1 + o(1))$ of conformal mapping at fixed point 0; $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The fixed point is called

attracting, if $0 < |\lambda| < 1$;

repelling, if $|\lambda| > 1$;

neutral, if $|\lambda| = 1$.

A neutral fixed point is called *parabolic*, if $\lambda = e^{2\pi i r}$, $r \in \mathbb{Q}$.

Theorem 2.11 *(Linearization Theorem). Every germ f at an attracting (repelling) fixed point is conformally conjugated to its linear part. More precisely, for every germ $f(z) = \lambda z + o(z)$, $|\lambda| \neq 0, 1$ there exists a unique conformal germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h \circ f(z) \equiv \lambda h(z)$ and $h'(0) = 1$.*

Addendum to Theorem 2.11. *Let in Theorem 2.11 $f = f_s(z)$ depend on an additional complex parameter $s \in V$ and $f_s(z)$ be holomorphic in $(s, z) \in V \times W$, here $V = V(0) \subset \mathbb{C}^n$ and $W = W(0) \subset \mathbb{C}$ are some neighborhoods of zero. Then the above normalized conjugating mapping $h = h_s(z)$ also depend holomorphically on s. More precisely, there exist (may be smaller) neighborhoods $V' = V'(0) \subset V$ and $W' = W'(0) \subset W$ such that $h_s(z)$ is holomorphic in $(s, z) \in V' \times W'$ and for every $s \in V'$ the mapping $h_s : W' \to \mathbb{C}$ is injective.*

Definition 2.12 The fixed point 0 of a germ $f(z) = cz^{k+1}(1 + o(1))$, $k \in \mathbb{N}$, $c \neq 0$ is called *super-attracting* (since its iterates converge to the constant mapping $z \mapsto 0$ super-exponentially.)

Theorem 2.13 *Every conformal germ $f(z) = cz^{k+1}(1 + o(1))$, $k \in \mathbb{N}$, $c \neq 0$ is conformally conjugated to the power $z \mapsto z^{k+1}$. That is, there exists a conformal germ $h : (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ (unique up to applying multiplication of the variable z by k-th root of unity) such that $f \circ h(z) \equiv h(z^{k+1})$.*

Definition 2.14 A periodic point of a rational function with period n is called *attracting (super-attracting, repelling, etc.)*, if it is a fixed point of the corresponding type for the n-th iterate R^n.

Corollary 2.15 *The (super-)attracting periodic points always lie in the Fatou set. The repelling periodic points always lie in the Julia set.*

Theorem 2.16 *The Julia set is closed, non-empty and perfect.*

Proof The closeness of the Julia set follows from definition. Let us prove that it is non-empty. Suppose the contrary: it is empty. Then all the iterates R^n form a normal family on the whole Riemann sphere. Therefore, there exists a uniformly convergent subsequence of iterates R^{n_k}. The limit should be a holomorphic self-mapping of the Riemann sphere (hence, a rational function of a finite degree m), and the iterates R^{n_k} with k large enough should have the same degree m. But this is impossible, since the iterates have exponentially growing degrees $deg(R^n) = d^n$, $d = degR$. The contradiction thus obtained proves that $J \neq \emptyset$.

Let us prove that J is perfect, that is, contains no isolated points. Let $I \subset J$ denote its discrete part: the subset of isolated points in J. One has $R^{-1}(I) = I = R(I)$, by definition. Let us consider the two following cases.

Case 1): I contains at least four points. Fix some four distinct points $A, B, C, D \in I$. Fix a point $a \in I$ and its neighborhood $V = V(a) \subset \overline{\mathbb{C}}$ containing no points of the Julia set different from a. Let us prove that the iterates R^n form a normal family on V. This would imply that $V \subset F$, hence $a \in F$. The contradiction thus obtained will prove the theorem.

For every $n \in \mathbb{N}$ one has $R^n(V) \cap J = \{R^n(a)\}$, by complete invariance of the Julia set. Therefore, given a sequence of iterates, passing to a subsequence one can achieve that $R^n(a)$ avoids three fixed values in I, say, A, B, C. This together with Montel Theorem implies normality of the iterates on V and proves the therorem.

Case 2): I is finite. Then the preimage of each point in I, being also a point in I, should be just one point, since the complete preimage $R^{-1}(I) = I$ should have the same cardinality, as I. Thus, each point in I is a critical value with the maximal branching order $d-1$, and the total number of critical points of the rational function R is no less than the cardinality $|I|$ of the set I times $d-1$. On the other hand. the number of critical points of a rational function of a given degree d equals $2d-2$. Hence, the set I consists of at most two points, and the preimage of each of them is one point that is a critical point of the maximal multiplicity $d-1$. Therefore, each point in I is a critical point with the same multiplicity, since $R^{-1}(I) = I$, and it is either a fixed point, or two-periodic point of the function R. In both subcases each point in I is a super-attracting periodic point, and hence, lies in F. Finally, $I \subset F$. The contradiction thus obtained proves the theorem. $\square$

2.3 Local dynamics at neutral periodic points and periodic components of the Fatou set

Definition 2.17 A component U of the Fatou set is said to be *n-periodic,* if $R^n(U) \subset U$: then $R^n(U) = U$.

Here we provide the classification of the periodic components of the Fatou set.

The following examples of periodic components contain (super-)attracting periodic points.

Schröder domain: a connected component U of the Fatou set containing an attracting periodic point z_0.

Böttcher domain: a connected component U of the Fatou set containing a super-attracting periodic point z_0.

Schröder (Böttcher) domains are also called the *immediate basin of attraction* of the corresponding periodic point z_0. This name is motivated by the following proposition.

Proposition 2.18 *The immediate basin of attraction of a (super-)attracting n-periodic point z_0 is the maximal connected open subset in $\overline{\mathbb{C}}$ containing z_0 where the iterates R^{mn} converge to the constant mapping $z \mapsto z_0$ uniformly on compact subsets, as $m \to \infty$.*

Proof Let us treat the case of attracting periodic point: the super-attracting case is treated analogously. The iterates R^{mn} converges to z_0 uniformly on a neighborhood of the point z_0, as do their linear parts $z \mapsto \lambda^m z$, $\lambda = (R^n)'(z_0)$, since R^n is conjugated to its linear part on a neighborhood of the point z_0. The family $R^{mn}|_U$ is normal, since $U \subset F$. Thus each sequence of iterates R^{mn} contains a subsequence converging uniformly on compact subsets in U. Each limit of a convergent subsequence of iterates $R^{mn}|_U$ is holomorphic, and hence, identically equal to z_0, by the previous statement, connectivity and uniqueness of analytic extension. This implies that the whole family R^{mn} converge uniformly on compact sets in U to z_0. The maximal connected open subset containing z_0 where the iterates R^{mn} converge to z_0 contains U and should be contained in the Fatou set, since the iterates are normal there, by convergence. Hence, it should coincide with U, by definition. The proposition is proved. $\square$

The above-mentioned Schröder and Böttcher domains are not the only examples of periodic components of the Fatou set. Two other types of periodic components are related to a class of neutral fixed points. To describe them, let us first describe the neutral points under question and the corresponding local dynamics.

The dynamics near a parabolic fixed point is described by the next theorem. Since its multiplier is a root of unity, passing to appropriate iterate one can achieve that the multiplier equals one.

Theorem 2.19 *For every given $k \in \mathbb{N}$ all the conformal germs $f(z) = z + cz^{k+1}(1 + o(1))$, $c \neq 0$ are homeomorphically conjugated between themselves.*

Example 2.20 The basic germ of parabolic mapping $z \mapsto z + z^{k+1}$ has $2k$ invariant rays $r_j = \{\arg z = \frac{\pi j}{k}\}$, $j = 0, \dots, 2k-1$, issued from the origin. The orbits in each ray r_j with even j go away from the origin: the rays r_j with even j are thus called the *repelling rays.* There exist a neighborhood $U = U(0) \subset \mathbb{C}$ of the origin where the restriction of the iterates f^m to every r_j with odd j converges to the origin: the rays r_j with odd j are thus called the *attracting rays*. Moreover, for every sector S with vertex at 0 whose closure does not contain repelling rays there exists a neighborhood $V = V(0) \subset \mathbb{C}$ such that all the iterates f^m are well-defined on $S \cap V$ and converge there uniformly to the constant mapping $z \mapsto 0$.

Corollary 2.21 *The parabolic periodic points always lie in the Julia set. For every parabolic periodic point z_0 of period n there exists a finite number of n-periodic connected components of the Fatou set where the iterates R^{mn} converge to the constant mapping $z \mapsto z_0$, as $m \to \infty$. More precisely, let $(R^n)'(z_0) = 1$, and $R(z) = z + c(z - z_0)^{k+1}(1 + o(1))$, as $z \to z_0$. Then the number of the above Fatou components equals k.*

Definition 2.22 The periodic components of the Fatou set from Corollary 2.21 associated to parabolic periodic points will be called the *Leau domains* or *immediate basins of attraction of parabolic periodic points.*

Example 2.23 Consider the family of quadratic polynomials $P_c(z) = z^2 + c$, $c \geq 0$. As was shown above, for $c = 0$ the Julia set is the unit circle. If $0 < c < \frac{1}{4}$, then the Julia set is a very wild Jordan curve that has no tangent lines at all. At the same time, we'll see later that it is a *quasicircle*: the image of a circle under a quasiconformal homeomorphism. In the interior component of the complement to the Julia set the iterates converge uniformly on compact subsets to the constant mapping sending everything to an attracting fixed point. In the exterior component the iterates converge to the constant mapping sending everything to the infinity. For $c = \frac{1}{4}$ the mapping P_c has a parabolic fixed point $z_0 = \frac{1}{2}$. Its Julia set is again a Jordan curve. But the restrictions of the iterates to the interior converges there to the latter parabolic point uniformly on compact subsets. Their restriction to the extertior converges to infinity. For $c > \frac{1}{4}$ the Julia set becomes totally disconnected: a (two-dimensional) Cantor set.

Definition 2.24 A number $\theta \in \mathbb{R}$ is *Diophantine,* if there exist $C, \varepsilon > 0$ such that for every rational number $\frac{p}{q}$ one has

$$|\theta - \frac{p}{q}| > \frac{C}{|q|^{2+\varepsilon}}.$$

Remark 2.25 The Diophantine numbers form a subset of complete Lebesque measure: the set of non-Diophantine numbers has measure zero. Each algebraic number is Diophantine.

Definition 2.26 A number $\theta \in \mathbb{R}$ is a *Bruno number,* if every germ of conformal mapping of the type $f(z) = e^{2\pi i\theta}z + o(z)$ at its fixed point 0 is conformally conjugate to its linear part: there exists a conformal germ $h(z) = z(1 + o(1))$ such that $h \circ f(z) = e^{2\pi i\theta}h(z)$.

Theorem 2.27 *(C.Sigel). Every Diophantine number is a Bruno number.*

Historical remark. A.D.Bruno have shown that if a number θ satisfies a certain condition on its continued fraction expansion, (Bruno Condition), then θ is a Bruno number. Bruno Condition is weaker than Diophantine property: each Diophantine number satisfies Bruno condition. A remarkable result of Jean-Christophe Yoccoz states that the converse is true: *θ is a Bruno number, if and only if it satisfies Bruno condition.* For this results and his other famous results in complex dynamics he got Fields medal at the International Congress of Mathematicians in Zürich in 1994.

Corollary 2.28 *Let R be a rational function with a n-periodic point z_0 such that $(R^n)'(z_0) = e^{2\pi i\theta}$, where θ is Diophantine (or more generally, a Bruno number). Then*

(i) $z_0 \in F$,

(ii) the connected component $U = U(z_0) \subset F$ of the Fatou set is simply connected and n-perioodic: $R^n(U) = U$,

(iii) there is a conformal isomorphism $h : U \to D_1$ conjugating R^n with the rotation by angle $2\pi\theta$: $h \circ R^n(z) = e^{2\pi i\theta} z$.

Definition 2.29 A component U of the Fatou set from the above corollary is called *Siegel disk.*

It appears that there is yet another possible type of periodic Fatou component. To describe it, let us recall the following definition and theorem.

Definition 2.30 [5, p. 104] Let $g : S^1 \to S^1$ be a circle homeomorphism. Fix some its lifting $G : \mathbb{R} \to \mathbb{R}$ to the universal covering and a point $x \in \mathbb{R}$. The limit

$$\rho = \lim_{m \to +\infty} \frac{G^m(x)}{m}$$

is called the *rotation number* of the homeomorphism g.

Theorem 2.31 *[5, p. 104] The above limit always exists. The rotation number is well-defined modulo $\mathbb{Z}$ and is independent on the choice of lifting G and point x.*

Theorem 2.32 *(V.I.Arnold, M.Herman)*[1] *Let $R : S^1 \to S^1$ be an analytic circle diffeomorphism with irrational Diophantine rotation number. Then it is conjugated analytically to a rotation.*

Corollary 2.33 *Let $S^1 \subset \overline{\mathbb{C}}$ be the unit circle. Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function such that there exists a $n \in \mathbb{N}$ for which $R^n(S^1) = S^1$. Let the restriction $R^n : S^1 \to S^1$ be a diffeomorphism with irrational Diophantine rotation number θ. Then $S^1 \subset F$ and the connected component U containing S^1 of the Fatou set satisfies the following statements:*

$(*)$ U is conformally equivalent to an annulus $A_r = \{r < |z| < 1\}$, $0 < r < 1$. Moreover, there exists a conformal isomorphism $h : U \to A_r$ such that $h \circ R^n \circ h^{-1}(z) = e^{2\pi i\theta} z$.

Definition 2.34 A n-periodic component of the Fatou set of a rational function R that satisfies the above statement $(*)$ is called *Arnold–Herman ring.*

Theorem 2.35 *Each periodic component of the Fatou set has one of the five following types:*

- *Schröder domain: the immediate attracting basin of an attracting periodic point;*
- *Böttcher domain: the immediate attracting basin of an attracting periodic point;*
- *Leau domain: an immediate attracting basin of a parabolic periodic point;*
- *Siegel disk;*
- *Arnold–Herman ring.*

The results of the next subsection will imply that the set of periodic components of the Fatou set (may be except for the Arnold–Herman rings) is always finite.

[1] V.I.Arnold proved this theorem for differmorphisms close to rotations and analytic on a (not too narrow) annulus containing the circle; the distance to rotations and the size of annulus both depend on the constants from the definition of Diophantine number

2.4 Critical orbits. Upper bound of the number of non-repelling periodic orbits

Definition 2.36 A *critical orbit* of a rational function R is the forward orbit $\{R^n(c) \mid n \geq 0\}$ of some its critical point c.

Theorem 2.37 *Each attracting periodic orbit attracts at least one critical orbit. In more detail, let a be a n-periodic attracting point. Then its immediate attracting basin contains a critical point of the iterate R^n (that is, a point eventually sent to a critical point of the function R by its appropriate iterate; thus its forward orbit converges to the periodic orbit of the point a).*

Proof Suppose the contrary. Let there exist a rational function R having an attracting periodic point a that does not satisfy the statement of the theorem. Without loss of generality we consider that it is a fixed point of the function R: one can achieve this by replacing R by its iterate R^n. Then the immediate attracting basin U of the attracting fixed point a contains no critical point of the function R. Set $\lambda = R'(a)$: $0 < |\lambda| < 1$. There exists a conformal mapping h of a neighborhood of the point a to a disk centered at 0 that conjugates R with its linear part $z \mapsto \lambda z$:

$$h \circ R(z) = \lambda h(z). \tag{2.1}$$

Let us show that h extends to a conformal isomorphism $h : U \to \mathbb{C}$. This implies that U is a once punctured Riemann sphere, and hence, the Julia set, which is contained in its complement, is reduced to at most one point. The contradiction thus obtained with perfectness of the Julia set will prove the theorem.

Fix a disk D_r contained in the image of the local mapping h. Set

$$V_0 = h^{-1}(D_r) \subset U, \ V_1 = R^{-1}(V_0) \cap U, \ V_2 = R^{-1}(V_1) \cap U \dots .$$

One has

$$V_0 \subset V_1 \subset V_2 \subset \dots = U.$$

Indeed, the disk D_r is invariant under the multiplication by λ, since $|\lambda| < 1$. Applying the conjugation, we get that $R(V_0) \subset V_0$. This implies that $R^{-1}(V_0) \supset V_0$ and hence, $V_1 \supset V_0$. The other above inclusions are proved analogously. The fact that the union of the domains V_j thus constructed is all of U follows from the definition of immediate attracting basin. Let us now extend the mapping h to V_j step by step.

The mapping $h = h_0$ is defined and biholomorphic on V_0. For every $z \in V_1$ set

$$h_1(z) = \lambda^{-1} h(R(z)).$$

One has $h_1 = h_0 = h$ on V_0, by (2.1), and equality (2.1) holds for thus extended mapping h on V_1. Similarly we set $h_2(z) = \lambda^{-1} h_1(R(z))$ for every $z \in V_2$ and get that $h_2 = h$ on V_1 etc. Finally we get a holomorphic mapping $h : U \to \mathbb{C}$. Note that for every j the mapping $R : V_{j+1} \to V_j$ is an unramified covering, and V_0 is simply connected. Therefore, $R : V_1 \to V_0$ is a conformal isomorphism, and consequently, so are all the mappings $R : V_{j+1} \to V_j$. This implies that the extended mapping $h : U \to \mathbb{C}$ thus constructed is a conformal isomorphism, and U is a once punctured Riemann sphere. This together with the above discussion proves the theorem. □

Corollary 2.38 *The total number N_{attr} of super-attracting and attracting periodic orbits of a rational function of a given degree d is no greater than the number $2d-2$ of its critical points.*

Let us now extend the above upper bound to the number N_{neut} of neutral periodic orbits.

Theorem 2.39 *One has*

$$N_{attr} + \frac{1}{2}N_{neut} \leq 2d - 2. \tag{2.2}$$

Proof

In what follows let $\mathcal{R}^d$ denote the space of all the rational functions of given degree d.

Claim 1. *For every function $R_0 \in \mathcal{R}^d$ there exists a neighborhood $V = V(R_0) \subset \mathcal{R}^d$ such that $N_{attr}(R) \geq N_{attr}(R_0)$ for every $R \in V$.*

Proof When we perturb a rational function, its attracting periodic orbits persist. Its super-attracting periodic orbit, if destroyed, generates at least one extra (super)attracting periodic orbit of the same period. Namely, let an iterate R^n have a super-attracting fixed point A. Then its restriction to some neighborhood $V = V(A)$ is a contracting mapping $V \to V$ in a local Euclidean metric. The latter property for a fixed V persists under perturbation and implies the existence of an attracting fixed point of the iterate R^n in V. This proves the claim. $\square$

Claim 2. *For every $R_0 \in \mathcal{R}^d$ there exists a function R arbitrarily close to R_0 for which*

$$N_{attr}(R) \geq N_{attr}(R_0) + [\frac{1}{2}(N_{neut}(R_0) + 1)]. \tag{2.3}$$

Proof Recall that the (super-) attracting periodic orbits persist under perturbation in the above sense, see the proof of Claim 1. Therefore, it suffices to show that perturbing R_0 one can transform at least half of its neutral periodic orbits to attracting ones, if N_{neut} is even (respectively, at least $\frac{N_{neut}+1}{2}$ orbits, if N_{neut} is odd).

Note that a n-periodic orbit is defined by the algebraic equation $R^n(z) = z$. The multiplier $\mu_n(R, z) = (R^n)'(z)$ is an analytic function of (R, z). Consider its restriction to the algebraic subset $\mathcal{P}_n = \{(R, z) \mid R^n(z) = z\} \subset \mathcal{R}^d \times \overline{\mathbb{C}}$. Note that the algebraic subset $\mathcal{P}_n$ is a hypersurface: it is defined by one equation. Hence, it has codimension one, and moreover, each its irreducible component has codimension one.[2]

Claim 3. *The restriction of the multiplier function μ_n to each irreducible component of the hypersurface $\mathcal{P}_n$ is not a constant with unit modulus.*

Proof Let $\mathcal{S}$ be an irreducible component of the set $\mathcal{P}_n$. It intersects each $\overline{\mathbb{C}}$-fiber of the product $\mathcal{R}^d \times \overline{\mathbb{C}}$ by a finite number of points. Hence, it is projected to all of $\mathcal{R}^d$, see the footnote. (In fact, this can be proved by using Weierstrass polynomials, without using the general Proper Mapping Theorem.) Suppose the contrary: the multiplier μ_n is a constant function on $\mathcal{S}$ and its value has modulus one. Then one can deform the pair (R, z) along the

[2] Recall that an algebraic set (i.e., a set defined by finite number of algebraic equations) is called *irreducible,* if it contains no smaller algebraic subset. It is well-known that each algebraic set is a finite union of irreducible ones, which are called its *irreducible components.* The algebraic version of the *Remmert Proper Mapping Theorem* states that the image of an algebraic set under a proper algebraic mapping is also algebraic. In our case it implies that every algebraic hypersurface in $\mathcal{R}^d \times \overline{\mathbb{C}}$ that intersects each $\overline{\mathbb{C}}$-fiber by a finite number of points is projected onto all of $\mathcal{R}^d$.

component $\mathcal{S}$ towards a pair $(\widetilde{R}, \widetilde{z})$, where $\widetilde{R}$ is arbitrary close to the monomial z^d and $\widetilde{z}$ is its neutral periodic orbit. This follows from epimorphicity of the projection $\mathcal{S} \to \mathcal{R}^d$. This implies that the monomial z^d also has a neutral n-periodic orbit, while it obviously doesn't: such an orbit should live on the invariant unit circle, while the derivative has modulus $|d|$ at the points of the unit circle. The contradiction thus obtained proves the claim. □

Corollary 2.40 *There exists a regular germ of analytic curve $\Gamma \subset \mathcal{R}^d$ through the point R_0 such that the multiplier functions μ_n corresponding to all the families of n-periodic orbits of functions $R \in \Gamma$ for all n induce multivalued holomorphic functions on Γ that are not constant functions with values of modulus one.*

Proposition 2.41 *Let $f : (\mathbb{C}, 0) \to \mathbb{C}$ be a germ of non-constant holomorphic function at 0 such that $|f(0)| = 1$. For every $\varepsilon > 0$ let $a(\varepsilon)$ denote the area (Lebesgue measure) of the intersection $f^{-1}(D_1) \cap D_\varepsilon$. One has*

$$\frac{a(\varepsilon)}{Area(D_\varepsilon)} \to \frac{1}{2}, \ \ as\ \varepsilon \to 0.$$

Proof In the case, when $f'(0) \neq 0$, the mapping f is asymptotically linear. The statement of the proposition for a linear mapping is obvious and easily implies the same statement in the general case (elementary asymptotics). In the opposite case, when $f(z) = cz^{k+1}(1+o(1))$, $c \neq 0$, the proof is similarly reduced (via elementary asymptotics) to the case, when $f(z) = z^{k+1}$. The preimage of the unit disk under the latter mapping looks asymptotically like the preimage of a half-plane, which is a union of sectors saturating a half-area. The proposition is proved. □

Fix an arbitrary finite set of neutral periodic orbits $O_1, \dots, O_N$ (their periods may be arbitrary). Let $\nu_1(R_0), \dots, \nu_N(R_0)$ denote the corresponding multipliers. The corresponding multivalued multiplier functions on Γ will be denoted by the same symbols $\nu_j(R)$. Without loss of generality we consider that the germs of restrictions $\nu_j|_\Gamma$ are single-valued: one can achieve this by replacing Γ by its appropriate covering ramified over the base point R_0. Let us introduce a local coordinate w on Γ centered at R_0. For every ε small enough and $j = 1, \dots, N$ set $W_{j,var} = D_\varepsilon \cap \nu_j^{-1}(D_1) \subset \Gamma$.

Claim 4. *For every ε small enough there exists a $z \in D_\varepsilon$ covered by at least $[\frac{N+1}{2}]$ sets $W_{j,\varepsilon}$.*

Proof Suppose the contrary: for every ε each $z \in D_\varepsilon$ is covered by at most $[\frac{N-1}{2}]$ sets $W_{j,\varepsilon}$. This implies that

$$\sum_j Area(W_{j,\varepsilon}) \leq [\frac{N-1}{2}] Area(D_\varepsilon) < \frac{N-1}{2} Area(D_\varepsilon). \tag{2.4}$$

On the other hand, $\frac{Area(W_{j,\varepsilon})}{Area(D_\varepsilon)} \to \frac{1}{2}$, as $\varepsilon \to 0$ for every j, by Proposition 2.41. Therefore, $\frac{\sum_j Area(W_{j,\varepsilon})}{Area(D_\varepsilon)} \to \frac{N}{2}$, as $\varepsilon \to 0$. Hence, for ε small enough the latter fraction under the limit will be greater than $\frac{N-1}{2} Area(D_\varepsilon)$, – a contradiction to inequality (2.4). The claim is proved. □

Claim 4 implies that for every given N neutral periodic orbits $O_1, \dots, O_N$ of the function R_0 there exist a function $R \in \Gamma$ arbitrarily close to R_0 and a collection of indices $j_1, \dots, j_k$, $k \geq [\frac{N+1}{2}]$ such that R has k attracting periodic orbits $\widetilde{O}_{j_l}(R)$, $l = 1, \dots, k$ that are arbitrarily close to the neutral orbits O_{j_l}. This proves Claim 2. $\square$

The left-hand side in inequality (2.3) is bounded from above by the number $2d-2$ (Corollary 2.38). This together with (2.3) implies Theorem 2.39. $\square$

Corollary 2.42 *The number of periodic components of the Fatou set (except may be for the Arnold–Herman rings) is finite.*

Proof Each periodic component of the Fatou set different from Arnold–Herman rings and Siegel disks is a basin of a (super-) attracting or parabolic periodic orbit. Each periodic orbit has a finite number of periodic basins. Each Siegel disk is associated to a neutral periodic orbit, and again each neutral orbit corresponds to at most a finite number of Siegel disk components. This together with uniform boundedness of the number of (super-) attracting and neutral periodic orbit (Theorem 2.39) proves the corollary. $\square$

2.5 Density of repelling periodic points in the Julia set

Theorem 2.43 *(G.Julia). The Julia set is the closure of the set of repelling periodic points.*

Proof It suffices to show that the union of all (not necessarily repelling) periodic points accumulates to all of J. This together with finiteness of the set of non-repelling periodic points (Theorem 2.39) and the fact that the repelling periodic points are contained in J will imply the statement of Theorem 2.43. Suppose the contrary: there exists an open subset $V \subset \overline{\mathbb{C}}$ that intersects the Julia set and contains no periodic points. Without loss of generality we consider that V is a disk that contains no critical points of the functions R and R^2. Let us show that the family of iterates $R^n|_V$ is normal: this would contradicts non-emptiness of the intersection $V \cap J$ and prove the theorem.

The preimage of the disk V under the mapping R (R^2) is a disjoint union of its d (respectively, d^2) copies that are bijectively sent onto V by the mapping under consideration. This follows from the choice of the disk V. Let us choose holomorphic inverses $f = R^{-1} : V \to \overline{\mathbb{C}}$, $g = R^{-2} : V \to \overline{\mathbb{C}}$ so that $f = R \circ g$. For every $z \in V$ and every $n \geq 1$ the values z, $f(z)$, $g(z)$, $R^n(z)$ are distinct. Indeed, if $f(z) = z$, then $z = R(z)$ is a fixed point of the mapping R, – a contradiction. Similarly, if $g(z) = z$, then $z = R^2(z)$ is a 2-periodic point. If $f(z) = R^n(z)$, then $z = R^{n+1}(z)$ is again a periodic point, etc. Finally, the family of iterates $R^n|_V$ takes values distinct from three distinct values of the holomorphic functions z, $f(z)$, $g(z)$. Hence, it is normal, by Holomorphic Variable Montel Theorem 2.6. This together with the above discussion proves Theorem 2.43. $\square$

2.6 Sullivan No Wandering Domain Theorem

Theorem 2.44 *(D.Sullivan). For every rational function with non-empty Fatou set each connected component of the Fatou set is pre-periodic. That is, for every component U of the Fatou set there exist $m, n \in \mathbb{N}$ such that $R^{m+n}(U) = R^m(U)$.*

In the proof of the theorem we use the following propositions.

Proposition 2.45 *Let R be a rational function, and let σ be a bounded measurable R-invariant almost complex structure on the Riemann sphere. Let $\Phi_\sigma : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiconformal homeomorphism transforming σ to the standard complex structure. Then the conjugate $R_\sigma = \Phi \circ R \circ \Phi^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map of the same degree, as R.*

Proof The mapping R_σ is locally quasiconformal outside a finite set, the image of the set critical points of the function R under the mapping Φ, since locally it is a composition of quasiconformal homeomorphisms. It fixes the standard complex structure σ_{st}: Φ^{-1} sends σ_{st} to σ, by definition; R preserves σ; Φ sends σ back to σ_{st}. Hence, the mapping R_σ is holomorphic outside a finite set (being locally a quasiconformal mapping preserving σ_{st} there), and it is continuous by construction. Hence, it extends holomorphically to the remaining finite set by Erasing Isolated Singularity Theorem. Finally, the mapping $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic everywhere, and hence, is rational, and the topological degrees of the mappings R and its conjugate R_σ are equal. The proposition is proved. □

Proposition 2.46 *Let R be a rational function, and let $(H_t)_{t\in[0,1]} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a continuous family of Riemann sphere homeomorphisms with $H_0 = Id$ that commute with R: $R \circ H_t = H_t \circ R$. Then each H_t fixes each point of the Julia set J: $H_t|_J \equiv Id|_J$.*

Proof For every $n \in \mathbb{N}$ the set P_n of n-periodic points is finite, and each H_t maps P_n to itself as a permutation. The latter permutation is identity, since $H_0 = Id$ and by continuity. The repelling periodic points being dense in the Julia set, $H_t \equiv Id$ there. □

The proof of Theorem 2.44 is done by contradiction. Suppose the contrary: there exists a wandering component U of the Fatou set. Then all its images $R^m(U)$ and preimages under all the iterates are pairwise disjoint. Using this we construct a family of R-invariant almost complex structures σ_s on $\overline{\mathbb{C}}$ depending on arbitrarily many parameters $s = (s_1, \dots, s_N) \in \mathbb{R}^N$. Here $N > 4d + 2 = dim_{\mathbb{R}}\mathcal{R}^d$, $\mathcal{R}^d$ being the space of rational functions of given degree $d = degR$. Afterwards we rectify the almost complex structures σ_s by quasiconformal homeomorphisms Φ_{σ_s}, and we get a family of rational functions $R_{\sigma_s} = \Phi_{\sigma_s} \circ R \circ \Phi_{\sigma_s}^{-1} \in \mathcal{R}^d$, by Proposition 2.45. Then a purely topological theorem implies that there exists a path $\alpha : [0,1] \to \mathbb{R}^N$ along which the function family R_{σ_s}, $s = \alpha(t)$ is constant. On the other hand we show that one can construct the complex structures σ_s so different that the family R_s be definitely non-constant along every continuous path. The contradiction thus obtained will prove Theorem 2.44.

Here for simplicity, to avoid details, we consider that U satisfies the following conditions:

(i) U is simply connected and the Riemann conformal mapping $h : D_1 \to U$ extends to a homeomorphism $h : \overline{D}_1 \to \overline{U}$;

(ii) for every $m \in \mathbb{N}$ the mapping $R^m : U \to R^m(U)$ is conformal (i.e., biholomorphic).

The proof of Theorem 2.44 will consist of the following steps.

Step 1. Construction of almost complex structure family σ_s, $s \in \mathbb{R}^N$. Fix an arbitrary almost complex structure family σ_s on U, $s \in \mathbb{R}^N$, with uniformly bounded dilatations. Let us extend it to $\mathcal{U} = \cup_{m\in\mathbb{Z}} R^m(U)$: push it forward to $R^m(U)$ by R^m for all $m \in \mathbb{N}$; pull it back to $R^{-k}(U)$ for all $k \in \mathbb{N}$ under the iterates R^k. Put σ_s to be the standard complex structure on the complement $\overline{\mathbb{C}} \setminus \mathcal{U}$. This yields a family of *measurable* almost complex structures σ_s

with uniformly bounded dilatations on the whole Riemann sphere that are well-defined and R-invariant. This follows from construction and disjointness of the sets $R^m(U)$, $m \in \mathbb{Z}$.

Step 2. Rectifying the almost complex structures thus obtained and topological arguments. For every s let $\Phi_{\sigma_s} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the quasiconformal homeomorphism sending σ_s to the standard complex structure and fixing the points 0, 1 and ∞. The functions $R_{\sigma_s} = \Phi_{\sigma_s} \circ R \circ \Phi_{\sigma_s}^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are rational of degree $d = degR$ (Proposition 2.45). They depend continuously on the parameter $s \in \mathbb{R}^N$, as do Φ_s (Theorem 1.11). This yield a continuous mapping $\mathbb{R}^N \to \mathcal{R}^d$, $s \mapsto R_{\sigma_s}$, and the real dimension of the image is less than N.

Theorem 2.47 *For every continuous mapping $F : W \to \mathbb{R}^m$ of a domain $W \subset \mathbb{R}^N$ with $N > m$ there exists a non-constant path $\alpha : [0,1] \to W$ along which $F = const$.*

We will use this theorem as known and we will not prove it here.

Theorem 2.47 together with the previous inequality implies that there exists a path $\alpha : [0,1] \to \mathbb{R}^N$ such that

$$R_{\sigma_{\alpha(t)}} \equiv R_{\sigma_{\alpha(0)}} \text{ for every } t \in [0,1]. \tag{2.5}$$

The mappings

$$H_t = \Phi^{-1}_{\sigma_{\alpha(0)}} \circ \Phi_{\sigma_{\alpha(t)}} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

are quasiconformal homeomorphisms, and they commute with R. Their quasiconformality follows from group property of the quasiconformal homeomorphisms. The commutation follows definition and (2.5).

Corollary 2.48 *The above homeomorphisms H_t fix the points of the Julia set of the function R: $H_t|_J \equiv Id$. Each of them sends the component U of the Fatou set to itself and transforms the almost complex structure $\sigma_{\alpha(t)}$ to $\sigma_{\alpha(0)}$.*

Proof The equality $H|_J \equiv Id$ follows from the continuity of the family H_t, the fact that $H_0 = Id$ by construction, their commutation with the function R and Proposition 2.46. Each H_t sends every component of the Fatou set to itself, since it fixes the points of its boundary. The transformation law for almost complex structures follows from definition. □

Step 3. Choosing σ_s to be essentially different almost complex structures, namely, with the following property:

$(*)$ *For every two distinct $s, s' \in \mathbb{R}^N$ there exists no quasiconformal homeomorphism $H_{s,s'} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that is identity on ∂U and that sends the almost complex structure $\sigma_s|_U$ to $\sigma_{s'}|_U$.*

This would imply that there exist no non-constant path $\alpha : [0,1] \to \mathbb{R}^N$ for which the corresponding family of homeomorphisms H_t satisfies the statement of Corollary 2.48. The contradiction thus obtained will prove Sullivan Theorem.

Definition 2.49 Two bounded almost complex structures σ and σ' on the unit disk D_1 are said to be *Teichmüller equivalent,* if there exists a homeomorphism $H : \overline{D}_1 \to \overline{D}_1$ that is quasiconformal on D_1 and identity on the boundary ∂D_1 that transforms σ to σ'.

Remark 2.50 The space of Teichmüller equivalence classes of all the bounded almost complex structures on D_1 is called the *universal Teichmüller space.*

Proposition 2.51 *For every $N \in \mathbb{N}$ there exists a N-dimensional family of pairwise Teichmüller non-equivalent almost complex structures on the unit disk.*

Proof Fix arbitrary three distinct points $A, B, C \in S^1 = \partial D_1$. Consider an arbitrary family of diffeomorphisms $(\phi_s)_{s\in\mathbb{R}^N} : \overline{D}_1 \to \overline{D}_1$ that fix them and that are pairwise distinct on the boundary: $\phi_s \not\equiv \phi_{s'}$ on S^1 for every $s \neq s'$. In addition we consider that their derivatives and those of their inverses are uniformly bounded on $\overline{D}_1$. For every s let σ_s be the image of the standard complex structure under the diffeomorphism ϕ_s. The almost complex structures σ_s thus constructed have uniformly bounded dilatations, as do the diffeomorphisms ϕ_s (by the assumption on boundedness of derivatives). We claim that σ_s are pairwise Teichmüller non-equivalent. Indeed, suppose the contrary: σ_s is Teichmüller equivalent to $\sigma_{s'}$, $s' \neq s$. That is, there exists a homeomorphism $h : \overline{D}_1 \to \overline{D}_1$ quasiconformal on the interior and identity on the boundary that transforms σ_s to $\sigma_{s'}$. Then the composition $H_{s,s'} = h_{s'}^{-1} \circ h \circ h_s$ preserves the standard complex structure and is quasiconformal on D_1, by construction. Hence, it is a conformal automorphism of the unit disk. On the other hand, it fixes the three points A, B, C on the boundary, by construction. Hence, it is identity. But h is also identity on the boundary. This implies that $h_s = h_{s'}$ on the boundary. The contradiction thus obtained proves the proposition. The proof of Sullivan Theorem under assumptions (i) and (ii) is complete. □

2.7 Hyperbolicity. Fatou Conjecture. The Mandelbrot set

We have already seen that each attracting periodic orbit attracts at least one critical orbit.

Definition 2.52 A rational function is said to be *hyperbolic,* if the orbit of each its critical point converges to a (super) attracting periodic orbit.

Remark 2.53 The set of hyperbolic rational functions is open in $\mathcal{R}^d$ for every d. Indeed, attracting periodic orbits (and convergence of orbits of given critical points to attracting orbits) persist under perturbation. Super-attracting periodic orbits either remain super-attracting, or become attracting after perturbation, and the critical orbits converging there (including the super-attracting periodic orbit under question) become converging to the newly born (super-)attracting orbit.

Example 2.54 A quadratic polynomial $P(z) = z^2 - \lambda z$ with $|\lambda| < 1$ is hyperbolic. Indeed, it has an attracting fixed point 0. Its critical points are the fixed point ∞ (hence, it is super-attracting) and the unique finite critical point $c = \frac{\lambda}{2}$. We know that the attracting basin of zero contains a critical point, and hence, the critical point c lies there. Finally, $P(z)$ is hyperbolic.

Conjecture 2.55 (Fatou) The set of hyperbolic rational functions is dense in $\mathcal{R}^d$ for every $d \geq 2$.

Fatou Conjecture is open even in the case of quadratic polynomials $P_c(z) = z^2 + c$. To discuss it in more detail, let us recall the definition of the *Mandelbrot set,* see Fig.1: this is the set

$$M = \{c \in \mathbb{C} \mid \text{the orbit } \{P_c^n(0)\}_{n\geq 1} \text{ is bounded}\}.$$

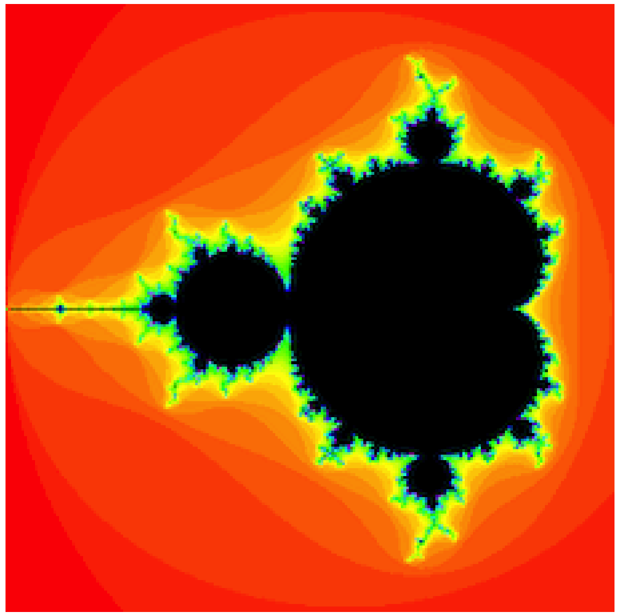

Figure 1: The Mandelbrot set.

Remark 2.56 The origin is the only finite critical point of every polynomial P_c. For every $c \notin M$ the polynomial P_c is hyperbolic, since $P_c^n(0) \to \infty$, as $n \to +\infty$: the orbit under question is unbounded, by definition, and hence, converges to ∞, since ∞ is a super-attracting fixed point.

Conjecture 2.57 (Quadratic Fatou Conjecture). The set of those parameters $c \in M$ for which P_c is hyperbolic is dense in M.

Conjecture 2.58 (MLC; A.Douady and J.Hubbard). The Mandelbrot set is locally connected.

Theorem 2.59 *The MCL Conjecture implies the Quadratic Fatou Conjecture.*

Theorem 2.59 will not be proved here.

In what follows we will discuss properties of the Mandelbrot set in more detail. It is known that the Mandelbrot set is a fractal set with beautiful structure. It is self-similar: it consists of infinitely many copies of itself.

Proposition 2.60 *The Mandelbrot set is contained in the closed disk* $\overline{D}_2 = \{|c| \leq 2\}$.

Proof Let $c \in \mathbb{C}$, $|c| = 2 + \varepsilon$, $\varepsilon > 0$. Let us show that $c \notin M$. To do this, it suffices to prove that $P_c^n(0) \to \infty$, as $n \to \infty$. One has $|P_c(0)| = |c| = 2 + \varepsilon$.

Claim. *For every* $z \in \mathbb{C}$ *with* $|z| \geq 2 + \varepsilon$ *one has*

$$|P_c(z)| \geq (1 + \varepsilon)|z|. \tag{2.6}$$

Proof One has $|z| \geq |c| = 2 + \varepsilon$, hence

$$|P_c(z)| = |z^2 + c| \geq |z^2| - |z| = (|z| - 1)|z| \geq (1 + \varepsilon)|z|.$$

□

The claim together with the inequality $|P_c(0)| \geq 2 + \varepsilon$ implies that

$$|P_c^n(0)| \geq (1 + \varepsilon)^{n-1}(2 + \varepsilon) \to \infty,$$

as $n \to \infty$, and proves the proposition. □

Definition 2.61 A connected component of the interior of the Mandelbrot set will be referred to, as a *component of the Mandelbrot set.* A component of the Mandelbrot set is *hyperbolic,* if each its point c corresponds to a hyperbolic polynomial P_c. A non-hyperbolic component (if it exists) is called a *queer component.*

Remark 2.62 The Quadratic Fatou Conjecture is equivalent to the conjecture saying that *each component of the Mandelbrot set is hyperbolic.* It is known that

- a component is queer, if and only if no its point corresponds to a hyperbolic polynomial.

- if a queer component U exists, then all the polynomials P_c with $c \in U$ are quasiconformally conjugate and $0 \in J(P_c)$;

- in the latter case for every $c \in U$ the corresponding Julia set $J(P_c)$ has positive area (Lebesgue measure) and there exists a measurable subset $X \subset J(P_c)$ of positive area (Lebesgue measure), $P_c^{\pm 1}(X) = X$, and a P_c-invariant measurable line field on X.

The Quadratic Fatou Conjecture is equivalent to the No Invariant Line Field Conjecture saying that *for every P_c with the Julia set of positive measure the above X and invariant line field on X do not exist.* This equivalence and No Invariant Line Field Conjecture will be discussed in more detail further on.

Remark 2.63 The period doubling bifurcations. The biggest visible component of the Mandelbrot set at Fig.1 is called the *main cardioid.* Let us denote it M_1. It consists exactly of those c for which P_c has an attracting fixed point. It is bounded by the algebraic curve consisting of those c for which P_c has a neutral fixed point. Its boundary intersects the real axis in two points. Their left intersection point is $c_1 = -\frac{3}{4}$. The corresponding polynomial P_{c_1} has a parabolic fixed point $z_1 = -\frac{1}{2}$ with multiplier -1. It can be seen as a 2-periodic point with multiplier 1. When we move c along the real axis to the left from the point c_1, the fixed point survives and becomes repelling. It appears that besides the latter repelling fixed orbit, the 2-periodic parabolic orbit of the polynomial P_{c_1} generates a 2-periodic attracting orbit of the polynomial P_c with $c < c_1$ close to c_1. This implies that the parameters $c < c_1$ close to c_1 lie in a hyperbolic component M_2 adjacent to M_1 that corresponds to polynomials with 2-periodic attracting orbits. Thus, the point $c_1 = -\frac{3}{4}$ corresponds to *period doubling bifurcation.* Now consider the left point c_2 of the intersection $\partial M_2 \cap \mathbb{R}$. It corresponds to the polynomial P_{c_2} with a parabolic 2-periodic orbit having multiplier -1. The latter orbit can be also considered as a 4-periodic parabolic orbit with multiplier equal to 1. When we move $c \in M_2$ to the left and cross c_2, then the 2-periodic parabolic orbit under question becomes repelling. It appears that besides the new repelling 2-periodic orbit, the above

4-periodic parabolic orbit of the polynomial P_{c_2} generates an attracting 4-periodic orbit of the polynomial P_c with $c < c_2$ close enough to c_2. This implies that $c < c_2$ close enough to c_2 belongs to a hyperbolic component adjacent to M_2 that corresponds to polynomials with a 4-periodic attracting orbit etc. Finally we get a chain of period doubling bifurcations corresponding to a chain of adjacent components $M_1, M_2, M_3, \dots$ separated by the bifurcation points c_j. The limit $c_F = \lim_{n\to\infty} c_n$ is called the *Feigenbaum point.*

Proposition 2.64 *Each component of the Mandelbrot set is simply connected.*

Proof The polynomials $\phi_n(c) = P_c^n(0)$ are uniformly bounded on M, more precisely, $|\phi_n(c)| \leq 8$. This follows from the fact that the iterates $P_c^n(z)$ with $|c| \leq 2$ (in particular, with $c \in M$) converge to infinity uniformly on the set $\{|z| \geq 8\}$, since for those z one has $|P_c(z)| \geq 8|z| - 2 \geq 6|z|$. The convergence is uniform in the parameter $c \in M$ for the same reason. Indeed, if $\phi_n(c) = P_c^n(0)$ lies outside the disk D_8 for some $c \in M$, then the iterates $P_c^{n+k}(0) = P_c^k(P_c^n(0))$ would converge to infinity, as $k \to \infty$, by the previous statement, – a contradiction to the assumption that $c \in M$. The uniform boundedness of the functions $\phi_n(c)$ on M together with the Maximum Principle implies their boundedness on every domain bounded by a Jordan curve contained in M. Hence, the latter domain is contained in M, by definition. This proves simple connectivity. □

The following remarkable result, which we state without proof, yields an explicit dynamical uniformization of every hyperbolic component of the Mandelbrot set.

Theorem 2.65 *(A.Douady and J.Hubbard). Let U be a hyperbolic component of the Mandelbrot set. For every $c \in U$ let $\mu(c)$ denote the multiplier of the corresponding periodic attracting orbit (for the least possible period). The mapping $\mu : U \to D_1$ is a conformal isomorphism.*

2.8 J-stability and structural stability: main theorems and conjecture

While Fatou conjecture on density of hyperbolic rational functions is completely open, the following remarkable classes of rational functions are known to be dense.

Definition 2.66 A rational function R_0 is *J-stable (structurally stable),* if there exist a neighborhood $V = V(R_0) \subset \mathcal{R}^d$ and a continuous family $(H_R)_{R\in V}$ of homeomorphisms $H_R : J_{R_0} \to J_R$ of the Julia sets (respectively, of the whole Riemann sphere), $H_{R_0} = Id$ on J_{R_0} (respectively, $\overline{\mathbb{C}}$) that conjugates R_0 to R:

$$H_R \circ R_0 \circ H_R^{-1} = R \text{ on } J_R \text{ (respectively, on } \overline{\mathbb{C}}).$$

Remark 2.67 Structurally stable rational functions are automatically J-stable. As it will be shown below, the converse is not true in general. The set of J-stable (structurally stable) rational functions is open, which follows by definition.

Theorem 2.68 *Structurally stable rational functions form a dense subset in $\mathcal{R}^d$, and thus, the same holds for the J-stable ones.*

Theorem 2.69 *The hyperbolic rational functions are J-stable.*

Conjecture 2.70 *J-stability is equivalent to hyperbolicity.*

This conjecture together with Theorem 2.68 would imply Fatou Conjecture.

In what follows we will prove the next criterium for a rational function to be J-stable, and then we'll prove the above theorems.

Let $\Pi \subset \mathcal{R}^d$ denote the closure of the set of the rational functions having parabolic periodic orbits. Set

$$\Sigma = \mathcal{R}^d \setminus \Pi.$$

Theorem 2.71 *The set Σ coincides with the set of J-stable functions.*

The proofs of the above theorems are based on holomorphic motions. The corresponding background material will be given in the two next subsections. Using holomorphic motions allows to prove quasiconformality of the conjugating homeomorphisms, which has important applications.

2.9 Holomorphic motions

Definition 2.72 Let V be a domain in a complex manifold with a marked point $O \in V$. A *holomorphic motion* of a subset $X \subset \overline{\mathbb{C}}$ over the domain V is a *disjoint* union of graphs $(\{w = \phi_z(t)\})_{z\in X}$ of holomorphic functions $\phi_z : V \to \overline{\mathbb{C}}$ with $\phi_z(O) = z$ for all $z \in X$. For every $t \in V$ we set

$$X_t = \{\phi_z(t) \mid z \in X\},\ X_O = X,\ H_{Ot}(z) := \phi_z(t).$$

The mapping $H_{Ot} : X \to X_t$ will be called the *holonomy* of the holomorphic motion.

Example 2.73 Let $V = \mathbb{H} = \{\operatorname{Im} t > 0\} \subset \mathbb{C}$. Set $O = i$. The graphs of the linear functions $\phi_z(t) = \operatorname{Re} z + t \operatorname{Im} z$ form a holomorphic motion of the whole Riemann sphere over V. Its holonomy is given by $\mathbb{R}$-linear isomorphisms $\mathbb{C} \to \mathbb{C}$.

We will use the following properties of holomorphic motions.

Lemma 2.74 *Every holomorphic motion of any subset $X \subset \overline{\mathbb{C}}$ extends to a holomorphic motion of its closure. Its holonomies are homeomorphisms.*

Proof We consider that the set X contains at least three distinct points A, B, C: otherwise it is finite and closed and we have nothing to prove. Then the family of functions ϕ_z is normal, by Holomorphic Variable Montel Theorem and since $\phi_z \neq \phi_G$ on V for every $G = A, B, C$ and $z \neq G$.

Claim 1. *For every $z_0 \in \overline{X} \setminus X$ the functions ϕ_z converge to a limit ϕ_{z_0} uniformly on compact subsets in V, as $z \to z_0$.*

Proof The family ϕ_z being normal, each their sequence contains a subsequence converging to a holomorphic limit uniformly on compact subsets in V. We have to show that any two converging sequences ϕ_{z_n}, ϕ_{w_n} with $z_n, w_n \to z_0$ have the same limit. Suppose the contrary: their limits ψ_1 and ψ_2 are distinct. One has $\psi_1(O) = \psi_2(O) = \lim_{n\to\infty} \phi_{z_n}(O) = \lim z_n = z_0$, by construction. Thus, ψ_1 and ψ_2 are distinct holomorphic functions with intersecting graphs. This implies that there exists a $N \in \mathbb{N}$ such that for every $n > N$ the functions ϕ_{z_n}, ϕ_{w_n} are distinct (hence, $z_n \neq w_n$) and their graphs are intersected. In more detail, restricting all the functions to appropriate holomorphic curve through O in V (e.g., a disk in a complex line) yields distinct functions ψ_j of one variable with intersected graphs. Therefore, the difference $\psi_2 - \psi_1$ has an isolated zero at O. Hence, the difference $\phi_{z_n} - \phi_{w_n}$, which limits to it, also has an isolated zero close to O, and the graphs of the functions ϕ_{z_n} and ϕ_{w_n} are intersected. But they cannot be intersected by definition of holomorphic motion. The contradiction thus obtained proves the claim. □

The limit functions ϕ_{z_0} from the claim taken for all $z_0 \in \overline{X} \setminus X$ extend the holomorphic motion from the set X to its closure. The graphs of functions ϕ_z with all $z \in \overline{X}$ are disjoint for distinct z, as in the above argument. Note that the statement and the proof of the claim remain valid also for every $z_0 \in X$. This implies that the functions $\phi_z(t)$ depend continuously on z in the topology of uniform convergence on compact subsets in V. This implies that the holonomies of the extended holomorphic motion are homeomorphisms. This proves the lemma. □

In the proof of quasiconformality of conjugating homeomorphisms for structurally stable rational functions we use the following general lemma on quasiconformality of holonomy of holomorphic motions. To state it, let us recall the following definitions.

For every collection of four distinct points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ consider the corresponding *cross-ratio*

$$\kappa[z_1, z_2, z_3, z_4] := \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}.$$

Remark 2.75 Recall that by definition, the mapping

$$z \mapsto \kappa[z_1, z_2, z_3, z]$$

is the conformal automorphism of the Riemann sphere that sends z_1, z_2, z_3 to 0, ∞ and 1 respectively. This implies that the cross-ratio as a function of four variables is invariant under the diagonal action of the conformal automorphism group $PSL_2(\mathbb{C})$ of the Riemann sphere.

Definition 2.76 Let $X, Y \subset \overline{\mathbb{C}}$ be closed subsets. A homeomorphism $f : X \to Y$ will be said *cross-ratio quasiconformal,* if for every $C > 1$ there exists a $K = K(C) > 1$ such that for every four points $z_1, \dots, z_4 \in X$ with

$$C^{-1} \leq |\kappa[z_1, z_2, z_3, z_4]| \leq C \tag{2.7}$$

one has

$$K^{-1}(C) \leq |\kappa[f(z_1), f(z_2), f(z_3), f(z_4)]| \leq K(C). \tag{2.8}$$

(The relation of cross-ratio quasiconformality to the usual quasiconformality will be explained in the next subsection.)

Lemma 2.77 (λ-lemma). *For every holomorphic motion of a closed subset $X \subset \overline{\mathbb{C}}$ over a domain V the holonomies H_{Ot} are cross-ratio quasiconformal. Moreover, for every compact subset $W \Subset V$ for every $C > 1$ there exists a $K = K_W(C) > 1$ such that for every $t \in W$ the mapping $f = H_{Ot}$ satisfies the conditions of the above definition.*

Proof Suppose the contrary. This means that there exist a compact subset $W \Subset V$, a $C > 1$ and sequences $t_n \in W$, $z_{jn} \in X$ with $j = 1, \dots, 4$ such that

$$C^{-1} \leq |\kappa[z_{1n}, z_{2n}, z_{3n}, z_{4n}]| \leq C \tag{2.9}$$

and the cross-ratio of the images $H_{Ot_n}(z_{jn})$, $j = 1, \dots, 4$ is unbounded from below or from above, say, from below. Passing to subsequences one can consider that $t_n \to t_0 \in W$, $z_{jn} \to z_j \in \overline{\mathbb{C}}$ and the latter cross-ratio tends to zero. For every n and every $t \in V$ let $\psi_n(t)$ denote the cross-ratio of the four points $H_{Ot}(z_{jn}) = \phi_{z_{jn}}(t)$, $j = 1, \dots, 4$. Then the previous cross-ratio of images equals $\psi_n(t_n)$. It tends to zero, and $t_n \to t_0$, as $n \to \infty$. The functions $\psi_n(t)$ are holomorphic on V and take values distinct from 0, 1 and ∞. This follows from the previous remark and since the values $\phi_{z_{jn}}(t)$, $j = 1, \dots, 4$ are distinct for every $t \in V$. Hence, the family ψ_n is normal (Montel Theorem), and its appropriate subsequence converges to a holomorphic limit $\psi(t)$ uniformly on compact subsets in V. Passing to the latter subsequence, we can consider that $\psi_n \to \psi$. One has $\psi(t_0) = 0$, since $\psi_n(t_n) \to 0$ and by uniform convergence on W. On the other hand, $\psi \not\equiv 0$, since $|\psi(O)| \geq C^{-1}$: the value $\psi(O)$ is the limit of the values $\psi_n(O)$, which are equal to the cross-ratios in (2.9), and hence, have absolute values no less than C^{-1}. Hence, there exists a $N \in \mathbb{N}$ such that for every $n \geq N$ the function ψ_n vanishes at some point τ_n converging to t_0, as in the above proof of Claim 1. This is equivalent to the statement that $\phi_{z_{4n}}(\tau_n) = \phi_{z_{1n}}(\tau_n)$. Thus, the functions $\phi_{z_{jn}}$ with $j = 1, 4$ have intersected graphs. This contradicts the definition of holomorphic motion and proves the lemma. $\square$

Lemma 2.78 *For every holomorphic motion of the whole Riemann sphere over a domain V the holonomies H_{Ot} are quasiconformal. Moreover, for every compact subset $W \Subset V$ there exists a $K = K_W > 1$ such that for every $t \in W$ the mapping H_{Ot} is K-quasiconformal.*

The lemma will be deduced from the λ-lemma in the next subsection.

Let us state the following very deep theorem, which is one of the main results in holomorphic motion theory. We will not use it and state it without proof.

Theorem 2.79 *(Z.Slodkowski). Each holomorphic motion of a subset $X \subset \overline{\mathbb{C}}$ over unit disk $V = D_1 \subset \mathbb{C}$ extends to a holomorphic motion of the whole Riemann sphere over unit disk.*

2.10 Quasiconformality: different definitions. Proof of Lemma 2.78

Definition 2.80 (Geometric quasiconformality). Consider a homeomorphism $f : U \to V$ of domains in $\overline{\mathbb{C}}$. Fix a point $z_0 \in U$ and consider a local holomorphic chart on a neighborhood of the point z_0. For every $z_1, z_2 \in U$ set

$$K(z_0, z_1, z_2) = |\frac{f(z_2) - f(z_0)}{f(z_1) - f(z_0)}|.$$

Set

$$K(z_0) = \overline{\lim}_{r \to 0} \max_{|z_1 - z_0| = |z_2 - z_0| = r} K(z_0, z_1, z_2).$$

Let $K > 1$. A homeomorphism f is said to be *K-quasiconformal,* if $K(z_0) \leq K$ for every $z_0 \in U$.

The following theorem is a deep analytic result, and we state it without proof.

Theorem 2.81 *The geometric K-quasiconformality is independent of choice of local chart. A homeomorphism is K-quasiconformal, if and only if it is geometrically K-quasiconformal in the sense of the above definition.*

Theorem 2.82 *A homeomorphism $f : U \to V$ of domains in $\overline{\mathbb{C}}$ that is cross-ratio quasiconformal (see Definition 2.76) is geometrically quasiconformal. In more detail, let $K(C)$ be the corresponding constant from Definition 2.76. Then f is geometrically $K(1+\varepsilon)$-quasiconformal for every $\varepsilon > 0$.*

Proof of Lemma 2.78. Consider a holomorphic motion of the whole Riemann sphere over a domain V. Fix a compact subset $W \Subset V$. Then for every $C > 1$ there exists a constant $K_W(C) > 1$ such that for every $t \in W$ the holonomy H_{Ot} of the holomorphic motion is cross-ratio quasiconformal so that inequality (2.7) implies inequality (2.8) with the constant $K_W(C)$. Therefore, it is geometrically $K = K_W(2)$-quasiconformal, by Theorem 2.82, and hence, K-quasiconformal in the sense of the usual definition, by Theorem 2.81. This proves the lemma. $\square$

Proof of Theorem 2.82. Fix two distinct points $z_0, w \in U$. One has

$$\kappa[z_0, w, z_1, z_2] \simeq \frac{z_2 - z_0}{z_1 - z_0}, \text{ as } z_1, z_2 \to z_0, \tag{2.10}$$

which follows from definition. In particular, the absolute value of the above cross-ratio tends to 1, as $z_1, z_2 \to z_0$ so that $z_1 - z_0 \simeq z_2 - z_0$. Let $K(C)$ be the family of quasiconformality constants from inequality (2.8) written for the mapping f. For every $\varepsilon > 0$

$$\kappa[f(z_0), f(w), f(z_1), f(z_2)] \leq K(1+\varepsilon) \tag{2.11}$$

whenever z_1 and z_2 are close enough to z_0 (dependently on ε) and their distances to z_0 are equal, by (2.10) and (2.8). On the other hand, the modulus of the cross-ratio from (2.11) is asymptotic to $|\frac{f(z_2)-f(z_0)}{f(z_1)-f(z_0)}| = K(z_0, z_1, z_2)$, as $z_1, z_2 \to z_0$, by asymptotics (2.10) applied to the images of the points under question. This implies that $K(z_0) \leq K(1+\varepsilon)$ for every $\varepsilon > 0$ and proves the theorem. $\square$

2.11 Characterization and density of J-stability

Here we prove Theorem 2.71 and density of the set of J-stable rational functions. Then we deduce Theorem 2.69. In the proof of Theorem 2.71 we use the two following propositions.

Proposition 2.83 *The open subset $\Sigma \subset \mathcal{R}^d$ is dense.*

Proof Recall that the complement $\Pi = \mathcal{R}^d \setminus \Sigma$ is the closure of the set of the rational functions having parabolic periodic orbits. Suppose the contrary: Σ is not dense, that is, the set Π has non-empty interior. Let $R \in Int(\Pi)$ be a function with a parabolic periodic orbit. Deforming it slightly, we can get a new function R_1 arbitrarily close to R so that the parabolic orbit of the function R deforms to an attracting periodic orbit O_1 of the function R_1, see the proof of Theorem 2.39. The function $R_1 \in Int(\Pi)$ is a limit of functions with parabolic periodic orbits. Take one of them, denote it by $\widetilde{R}_1$. It has both an attracting periodic orbit close to O_1 and a parabolic periodic orbit. Then we can deform $\widetilde{R}_1$ to a rational function $R_2 \in Int(\Pi)$ so that the attracting periodic orbit persists and the parabolic orbit of the function $\widetilde{R}_1$ generates a new attracting periodic orbit of the function R_2. Thus, the function R_2 has at least two attracting periodic orbits. Continuing this procedure would yield a rational function with arbitrarily large number of attracting periodic orbits. The contradiction thus obtained to Corollary 2.38 proves the proposition. □

Proposition 2.84 *The rational functions from the set Σ have no neutral periodic orbits.*

Proof Let R be a rational function with a neutral periodic orbit. Then $R \in \Pi$. Indeed, in the case, when the orbit is parabolic, this is obvious. In the opposite case the orbit under question has multiplier different from one and hence, persists under small deformations. It depends holomorphically on the coefficients of the underlying rational function together with its multiplier (Implicit Function Theorem). The multiplier is a non-constant holomorphic function of the coefficients, as in the proof of Theorem 2.39. Hence, one can achieve that the multiplier under question be a root of unity by arbitrarily small perturbation of the function R. Thus, R can be approximated arbitrarily well by functions with parabolic orbits. Hence, $R \in \Pi$. The proposition is proved. □

Proof of Theorem 2.71. Let $R_0 \in \Sigma$. Let us prove that it is J-stable. To do this, we consider the mapping

$$\rho : V \times \overline{\mathbb{C}} \to V \times \overline{\mathbb{C}}, \ \rho(R, z) := (R, R(z)). \tag{2.12}$$

Proposition 2.85 *For every simply connected neighborhood $V = V(R_0) \subset \Sigma$ there exists a ρ-invariant holomorphic motion of the Julia set $J(R_0)$ over V.*

Proof The repelling periodic points are dense in the Julia set of the function R_0. When we deform R_0 in its neighborhood V, the repelling periodic points persist, remain repelling and are holomorphic functions of the coefficients of variable rational function $R \in V$. This follows from Proposition 2.84 and the implicit function theorem. Their graphs thus form a holomorphic motion over V. It extends to a holomorphic motion of the whole Julia set, by density and Lemma 2.74. The holomorphic motion of the repelling periodic points is ρ-invariant, since the set of periodic points of a given period is finite and the underlying rational function permutes them in the same way, as does R_0, by continuity. Therefore, its extension to the Julia set by passing to limits is also ρ-invariant. The proposition is proved. □

The holonomy $H_R := H_{R_0 R} : J(R_0) \to \overline{\mathbb{C}}$ of every ρ-invariant holomorphic motion of the Julia set $J(R_0)$ is a conjugacy between R_0 and R on $J(R_0)$ and its image $H_R(J(R_0))$, by definition and ρ-invariance. One has $H_{R_0} = Id$ by definition. The mapping H_R sends the Julia set $J(R_0)$ homeomorphically onto $J(R)$. Indeed, it sends repelling periodic orbits to repelling periodic orbits, as in the above argument, hence $H_R(J(R_0)) \subset J(R)$. The opposite inclusion $H_R^{-1}(J(R)) \subset J(R_0)$ is proved analogously with interchanging the roles of the functions R_0 and R. Thus, the mapping R_0 is J-stable.

Let us prove the converse: every J-stable rational function R lies in Σ. Indeed, suppose the contrary: there exists a J-stable rational function in Π. Then its whole neighborhood consists of J-stable functions, by openness of J-stability condition. Hence, there exists a J-stable function R_0 with a parabolic periodic point A. Perturbing R_0, we can get a J-stable rational function R_1 having a neutral periodic point B close to A with multiplier different from one. Note that when we deform R_1 in its arbitrarily small neighborhood $V = V(R_1)$, the periodic point $B = B(R_1)$ is included into a holomorprhic family of periodic points $B(R)$ of variable function R with the same period, as $B(R_1)$, and variable holomorphic multiplier $\mu = \mu(R) \not\equiv const$. The multiplier function has modulus greater than one on some open subset $U \subset V$ and less than one on another open subset $W = V \setminus \overline{U}$. Thus, the periodic point $B(R)$ lies in the Julia set for $R \in U$ and in the Fatou set for $R \in W$. Hence, as R goes from the open set U to its exterior W, the periodic point $B(R)$ escapes from the Julia set to the Fatou set. The conjugacy $H_{R_1} \circ H_R^{-1}$ between R_1 and R is forced to send $B(R)$ to $B(R_1)$ for $R \in U$, by continuity and periodicity. Hence, $B(R_1)$ lies in the Julia set, as does $B(R)$ for $R \in U$. Analogously we get that $H_R \circ H_{R_1}^{-1}(B(R_1)) = B(R)$ for all R close to R_1, by continuity and periodicity, and hence, $B(R) \in J(R)$ for $R \in W$. But $B(R) \in F(R)$ for $R \in W$, by construction. The contradiction thus obtained proves Theorem 2.71. $\square$

Proof of Theorem 2.69. Suppose the contrary: there exists a hyperbolic rational function R that is not J-stable. That is, $R \in \Pi$, hence, R is a limit of parabolic functions: functions with parabolic periodic orbits. Hyperbolicity being an open condition, we can and will consider that R itself is parabolic. One can deform R to an arbitrarily close function R_1 so that the parabolic periodic orbit of the function R generates an additional attracting periodic orbit of the function R_1. The latter new orbit attracts no critical orbit of the function R_1. Indeed, under the above deformation the attracting periodic orbits of the function R persist (some super-attracting ones may become attracting), together with the critical orbits attracted to them, and become attracting orbits of the function R_1. All the critical orbits of the function R converge to its attracting orbits, by hyperbolicity. Hence, the same remains valid for the function R_1 and the deformed attracting orbits. Thus, the new born additional attracting orbit can attract no critical orbit. The contradiction thus obtained to Theorem 2.37 proves Theorem 2.69. $\square$

Theorem 2.86 *(R.Mañe [20]). A J-stable rational function cannot have Arnold–Herman rings.*

This is a tricky theorem, and we will not give its proof here.

Corollary 2.87 *The only possible types of periodic Fatou components of a J-stable rational function are immediate attracting basins of (super-) attracting periodic points.*

Proof A J-stable rational function R cannot have neutral periodic orbits, by Theorem 2.71 and Proposition 2.84. Hence, it can have neither Leau domains, nor Siegel disks. It has no Arnold-Herman rings, by Mañe's Theorem 2.86. This together with Theorem 2.35 on the classification of periodic Fatou components proves the corollary. $\square$

2.12 Characterization and density of structural stability

Recall that R is J-stable, if and only if $R \in \Sigma$. Let $\Sigma' \subset \Sigma$ denote the subset of the rational functions $R \in \Sigma$ satisfying the following conditions:

(i) The critical points of the function R are simple (thus, R has $2d-2$ distinct critical points);

(ii) The forward orbits of the critical points are infinite, i.e., not pre-periodic.

(iii) For every distinct critical points $c_1, c_2 \in F$ and every $m, n \in \mathbb{Z}_{\geq 0}$ one has $R^m(c_1) \neq R^n(c_2)$.

Theorem 2.88 *The set Σ' is open and dense in $\mathcal{R}^d$.*

Theorem 2.89 *The set Σ' coincides with the set of structurally stable rational functions.*

The two above theorems together imply Theorem 2.68.

We will also prove the following theorem having important applications. One of them is the reduction of the Fatou conjecture to a purely ergodic conjecture on the dynamics on the Julia set (No Invariant Line Field Conjecture). This conjecture and the proof of the theorem will be presented in the next subsection.

Theorem 2.90 *Every two rational functions lying in the same connected component of the set Σ' are quasiconformally conjugated.*

Proof of Theorem 2.88. The set Σ being open, it contains an open and dense subset of those rational functions whose critical points are simple and whose critical values are distinct.

Remark 2.91 When we deform a rational function inside the set Σ, its critical points in the Fatou set do not escape to the Julia set and vice versa. This follows from conjugacy of the deformed functions on their Julia sets: the points $z \in J$ where the germ of the underlying function R is not a germ of local homeomorphism $(J, z) \to (J, z)$ are exactly the critical points of the function R lying in J, and their number should remain constant, by conjugacy.

Let $R_0 \in \mathcal{R}^d$ be a rational function with a pair of simple critical points c_1 and c_2. For every $R \in \mathcal{R}^d$ close to R_0 the critical points persist and are holomorphic functions $c_j = c_j(R)$ in the coefficients of the variable function R.

Proposition 2.92 *No relation $R^m(c_1(R)) = R^n(c_2(R))$ with $m, n \in \mathbb{Z}_{\geq 0}$ holds locally identically in R.*

Proof Suppose the contrary: a relation $R^m(c_1) = R^n(c_2)$ holds locally identically in a neighborhood of a rational function R_0. The latter relation being algebraic, it holds (extends analytically) on a neighborhood of any path in $\mathcal{R}^d$. In particular, along a path connecting a given triple (R, c_1, c_2) to another one, for which the critical values $R(c_1(R))$ and $R(c_2(R))$ are distinct. Afterwards applying appropriate continuous families of conformal automorphisms of the Riemann sphere from the left and from the right, one can deform the triple thus obtained to a triple (R, c_1, c_2) with $c_1 = R(c_1) = 0$, $c_2 = R(c_2) = \infty$; the above relation should remain valid for the deformed triples. Then the critical points 0 and ∞ of the new rational function are fixed and hence, have disjoint fixed orbits. Hence, the above relation does not hold for the deformed function. The contradiction thus obtained proves the proposition. $\square$

Proposition 2.93 *No relation $R^{m+n}(c) = R^m(c)$ with c being a critical point of the function R, $m \geq 0$, $n \geq 1$ holds locally identically in R.*

Proof Suppose the contrary, a relation from the proposition holds locally identically near a given pair (R, c) of a rational function and its simple critical point. One can deform it to a similar pair (R_1, c_1) with a hyperbolic rational function R_1 that has no super-attracting periodic orbit; the relation should remain valid for the deformed pairs, as in the above proof. But this is impossible, since each critical orbit of the function R_1 converges to an attracting periodic orbit and hence, is infinite. The contradiction thus obtained proves the proposition. $\square$

The two above propositions together imply that Σ' is dense. Let us show that it is open.

Fix an $R \in \Sigma'$. Let us show that deforming R cannot create new relations of type (iii) for functions arbitrarily close to R. Note that a priori, the latter relations can be created only for the critical points lying in the Fatou set: all the functions close to R are conjugate to $R|_J$ on their Julia sets, and hence, on the union of the critical orbits lying in the Julia sets, see Remark 2.91. Each critical orbit of the function R lying in the Fatou set converges to an attracting periodic orbit, since $R \in \Sigma$ and R has no periodic critical orbits (i.e., super-attracting orbits) by assumption. Recall that the number of attracting orbits is finite. There is a neighborhood W of the union of the attracting orbits such that $R|_W$ is injective and W is strictly invariant: $R(W) \Subset W$. The union of all the critical orbits, except for its finite subset, lies in W, where the dynamics is bijective. The attracting orbits and their strictly invariant neighborhood W persist under deformations of the function R. Therefore, no relation of type (iii) can be born for the parts of the critical orbits lying in W, by its invariance and bijectivity of the dynamics in W. No new relation (iii) can be born for the finite part of the union of critical orbits under small deformation of the function R for obvious reasons (finiteness). This proves openness of the set Σ' and Theorem 2.88. $\square$

Proof of Theorem 2.89. Each structurally stable rational function lies in Σ'. Indeed, it is J-stable, which follows from definition, and hence, lies in Σ. It cannot satisfy locally non-identical relations $R^m(c_1) = R^n(c_2)$ or $R^{m+n}(c) = R^m(c)$ on critical orbits for obvious reasons: a continuous family of conjugacies should map critical orbits to critical orbits. Hence, it lies in Σ'. Let us proof the converse (the main part of the proof). The idea of proof is analogous to that of the proof of the analogous theorem for J-stablility from the previous subsection. Fix a rational function $R_0 \in \Sigma'$. Let us prove that it is structurally stable. To do this, fix its small contractible neighborhood $V = V(R_0) \subset \Sigma'$ and consider the mapping $\rho : V \times \overline{\mathbb{C}}$, $\rho(R, z) = (R, R(z))$ from the previous subsection. For V chosen to be small enough we construct a ρ-invariant holomorphic motion of the whole Riemann sphere over V. Then the corresponding holomomies $H_R = H_{R_0 R}$ will form a continuous (and even holomorphic) family of conjugacies between R_0 and $R \in V$. This will prove structural stability.

We have already a ρ-invariant holomorphic motion of the Julia set $J = J(R_0)$ constructed in the previous subsection. Let us construct a ρ-invariant holomorphic motion of the Fatou set. The union of these two holomorphic motions of the Fatou and Julia sets will yield a ρ-invariant motion of the Riemann sphere we are looking for. We consider that $J \neq \overline{\mathbb{C}}$: otherwise, the holomorphic motion is already constructed. Then each periodic component of the Fatou set of the function R_0 is the immediate attracting basin of an attracting periodic point, by Corollary 2.87.

Step 1. A preliminary construction of ρ-invariant holomorphic motion of neighborhoods of attracting periodic orbits. Recall that the attracting periodic points of the function R_0 persist under perturbation and generate holomorphic families of attracting periodic points of functions close enough to R_0. We consider that they are well-defined for all $R \in V$ and depend holomorphically on them, taking V small enough. Fix a family $A(R)$ of n-periodic attracting points, $R_0^n(A(R)) = A(R)$, set $\mu = \mu(R) = (R^n)'(A(R))$. The multiplier μ also depends holomorphically on R and takes values in the unit disk punctured at 0. There exist a neighborhood $W = W(A(R_0)) \subset \overline{\mathbb{C}}$ and a holomorphic family of mappings $h_R : W \to \mathbb{C}$, $h_R(A(R)) = 0$, that conjugate R^n to the linear mappings $z \mapsto \mu(R)z$: $h_R \circ R = \mu(R)h_R$. The mappings h_R being holomorphic in $(R, w) \in V \times W$. (This will hold after choosing V small enough.) This follows from the Addendum to the Linearization Theorem 2.11. In addition, we can and will choose W to be strictly invariant under the mapping R_0^n, $R_0^n(W) \Subset W$, and the same statement holds for all $R \in V$.

Consider now the holomorphic family of the corresponding linear mappings as the mapping $\lambda : V \times \mathbb{C} \to V \times \mathbb{C}$, $\lambda(R, z) = (R, \mu(R)z)$. Consider the exponential universal cover $\mathbb{C} \to \mathbb{C}^*$, $w \mapsto e^{2\pi i w}$ over the $\mathbb{C}$-fibers punctured at the origin. This is a homomorphism from the additive group $\mathbb{C}$ to the multiplicative group $\mathbb{C}^*$ whose kernel is the subgroup $\mathbb{Z} \subset \mathbb{C}$. This lifts λ to the mapping

$$\widetilde{\lambda} : V \times \mathbb{C} \to V \times \mathbb{C},\ \widetilde{\lambda}(R, w) = (R, w + \nu(R)),\ \nu(R) = \frac{\ln \mu(R)}{2\pi i},$$

$$\operatorname{Im}(\nu(R)) = -\frac{1}{2\pi} \operatorname{Re}(\ln \mu(R)) > 0.$$

Let us represent each $w \in \mathbb{C}$ as a real linear combination of the complex numbers 1 and $\nu(R_0)$ viewed as vectors in $\mathbb{R}^2 = \mathbb{C}$:

$$w = s_1(w) + s_2(w)\nu(R_0), \quad s_1(w), s_2(w) \in \mathbb{R}.$$

The next claim follows immediately from construction.

Claim 1. *The functions $\Phi_w(R) = s_1(w) + s_2(w)\nu(R)$ form a holomorphic motion of the complex line $\mathbb{C}$ over V that is invariant under the mapping $\widetilde{\lambda}$ and the integer translations of the $\mathbb{C}$-fibers.*

Corollary 2.94 *The functions $\phi_z(R) = e^{2\pi i \Phi_w(R)}$ with $w = \frac{\ln z}{2\pi i}$ are well-defined (independent on choice of branch of logarithm) and form a λ-invariant holomorphic motion of the punctured line $\mathbb{C}^*$ over V. It is extended to a holomorphic motion of the line $\mathbb{C}$ by the zero section $\phi_0(R) \equiv 0$.*

Recall that the mapping $H : (R, z) \mapsto (R, h_R(z))$ conjugates the mapping $\rho^n(R, z) = (R, R^n(z))$ to the mapping λ. Therefore, the inverse H^{-1} transforms the λ-invariant holomorphic motion $\phi_z(R)$ to a ρ^n-invariant holomorphic motion $\psi_z(R) = h_R^{-1}(\phi_z(R))$ of a neighborhood $W = W(A(R_0))$ over V. Applying the iterates ρ^j, $j = 1, \dots, n-1$ extends it to a ρ-invariant holomorphic motion of the union of neighborhoods of the points $R_0^j(A(R_0))$; it is defined by the functions $\psi_{R^j(z)}(R) := R^j(\psi_z(R))$, $z \in W$. This is a holomorphic motion of a neighborhood of the periodic orbit of the point $A(R_0)$.

Step 2. Correcting thus constructed holomorphic motion of neighborhoods of attracting orbits to take into account the critical orbits attracted to it. The conjugacies between the functions R_0 and R we are looking for should map critical orbits to critical orbits. In other terms, for every $z = R_0^k(c)$, c being a critical point of the function R_0 the graph of the corresponding function $\psi_z(R)$ from the holomorphic motion should form a family of postcritical values $R^k(c(R))$ of the functions $R \in V$. On the other hand, the holomorphic motion constructed above does not have this property automatically. Let us correct it to respect the critical orbits. To do this, fix a small enough fundamental annulus domain B for the linearized mapping $h_{R_0} \circ R_0^n \circ h_{R_0}^{-1}(z) = \mu(R_0)z$. Then $B' = h_R^{-1}(B)$ is a fundamental domain with compact closure for the mapping R_0^n. Let us mark all the postcritical points in B', i.e., points lying in the critical orbits of the function R_0: this yields a finite subset $\{C_1(R_0), \dots, C_k(R_0)\} \subset B'$, $C_j(R_0) = R_0^{n_j}(c_j)$, c_j being some critical points of the function R_0. We adjust the annulus B so that no point $C_j(R_0)$ lies in the boundary $\partial B'$. The critical and hence, postcritical points of the function R_0 are included into holomorphic families of (post)critical points of all $R \in V$: $C_j = C_j(R)$ denote the holomorphic families of postcritical points generated by $C_j(R_0)$.

Consider the above exponential universal cover $\mathbb{C} \to \mathbb{C}^*$ and the lattice $L_{R_0} =< 1, \nu(R_0) >$ generated by 1 and the number $\nu(R_0)$. (Note that the quotient $\mathbb{T}^2 = \mathbb{C}/L_{R_0}$ is the complex torus, the orbit space of the linearized mapping $z \mapsto \mu(R_0)z$. Its modulus equals $\nu(R_0)$.) Fix a fundamental parallelogram $\widetilde{B} \subset \mathbb{C}$ projected to the above fundamental annulus B. Let $\widetilde{C}_j(R)$ denote the liftings to $\widetilde{B}$ of the images $h_R(C_j(R)) \in B$ of the above postcritical points. Fix a collection of disjoint closed disks $\Delta_j \Subset \widetilde{B}$ centered at $s_j = \widetilde{C}_j(R_0)$, $j = 1, \dots, k$. Taking V small enough, we will consider that $\widetilde{C}_j(R) \in Int(\Delta_j)$ for all j. Let us correct the above-constructed linear holomorphic motion $\Phi_w(R) = s_1(w) + s_2(w)\nu(R)$ inside the disks Δ_j as follows. We have a prescribed holomorphic motion $\widetilde{C}_j(R)$ of their centers s_j and the prescribed motion $\Phi_w(R)$ of their boundaries. Each point $w \in \Delta_j \setminus \{s_j\}$ has a unique representation as a convex combination

$$w = p(w)s_j + (1 - p(w))t(w), \ t(w) \in \partial\Delta_j, \ 0 \le p(w) < 1. \tag{2.13}$$

For every $w \in \Delta_j$ set

$$\Phi_w(R) = p(w)\widetilde{C}_j(R) + (1 - p(w))\Phi_{t(w)}(R).$$

The functions $\Phi_w(R)$ thus constructed extend the holomorphic motion Φ_w of the exterior of the disk Δ_j to its interior. Indeed, their holomorphicity is obvious. Note that the holomomy of the linear holomorphic motion Φ_w of the boundary $\partial\Delta_j$ is real-linear, and hence, transforms the boundary $\partial\Delta_j$ to a strictly convex curve for every R. Hence, the graphs of the functions Φ_w, including the new ones, are pairwise disjoint, by construction. In more detail, the uniqueness of convex combination representation (2.13) remains valid if we replace the pair (Δ_j, s_j) by another pair consisting of any strictly convex domain and any point in its interior.

Now let us make the above corrections in all the disks Δ_j: we get a corrected holomorphic motion of the fundamental parallelogram $\widetilde{B}$ that is unchanged on its boundary. Let us extend it by translations by the lattices L_R to a holomorphic motion of the whole line $\mathbb{C}$ that is invariant under the family of lattice translations, i.e., invariant under mapping $\widetilde{\lambda}$ and the integer translations of the $\mathbb{C}$-fibers. The holomorphic motion thus obtained will be now denoted by Φ_w. Let ϕ_z denote the corresponding holomorphic motion from the above corollary. It is transformed by family of inverse conjugating mappings h_R^{-1} to a ρ^n-invariant holomorphic motion $\psi_z(R)$. The latter extends to a ρ-invariant holomorphic motion (also denoted by ψ_z) of an invariant neighborhood of the attracting orbit under question, as in Step 1.

Step 3. Extension of thus constructed local holomorphic motions by dynamics. Let us do the above construction of ρ-invariant holomorphic motions (respecting post-critical points) in neighborhoods of all the attracting periodic orbits of the function R_0. We consider that all the latter neighborhoods are strictly R_0-invariant, and this remains valid for all $R \in V$, taking V small enough. Next we extend the ρ-invariant holomorphic motion thus constructed of the union of the above neighborhoods by taking pullbacks of the corresponding graphs of holomorphic functions under the mappings ρ^n. The preimage of each graph is a disjoint union of graphs of holomorphic functions, since the critical values of each iterate of the mapping ρ form a finite union of graphs of functions $\psi_z(R)$: the holomorphic motions are constructed to respect the postcritical points. Note that the R-orbit of each point in the Fatou set of the function $R \in V$ converges to an attracting periodic orbit: each component of the Fatou set is eventually mapped to a periodic component (Sullivan No Wanderind Domain Theorem), and each periodic component is an immediate attracting basin. This implies that taking ρ^n-pullbacks for all $n \in \mathbb{N}$ of the local holomorphic motions in neighborhoods of the attracting periodic orbits extend them to a global ρ-invariant holomorphic motion of the Fatou set. This extends the holomorphic motion of the Julia set constructed in the previous subsection to a ρ-invariant holomorphic motion of the whole Riemann sphere. Theorem 2.89 is proved. □

2.13 Proof of Theorem 2.90

Proof of Theorem 2.90. Fix a $R_0 \in \Sigma'$. We already know that there exist a neighborhood $V = V(R_0) \subset \Sigma'$ and a ρ-invariant holomorphic motion over V, and its holonomies $H_R = H_{R_0,R}$ conjugate R_0 to R on the whole Riemann sphere. The holonomies H_R are quasiconformal for every $R \in V$, by Lemma 2.78. Now fix a function R_1 lying in the same connected component of the set Σ', as R_0, and a path $\alpha : [0,1] \to \Sigma'$, $\alpha(0) = R_0$, $\alpha(1) = R_1$. Let us show that R_0 and R_1 are quasiconformally conjugated. For every point $Q \in \alpha[0,1]$ take its neighborhood $V(Q)$ such that all the rational functions $R \in V(Q)$ are quasiconformally conjugated to Q. One can cover α by a finite set of the latter neighborhoods. This implies that there exists a chain of points $Q_j \in \alpha[0,1]$, $Q_0 = R_0, Q_1, \dots, Q_n = R_1$ such that Q_{j+1} and Q_j are conjugated by quasiconformal homeomorphisms $h_j : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$: $Q_{j+1} = h_j \circ Q_j \circ h_j^{-1}$. Then their composition $h_{n-1} \circ \dots \circ h_0$ conjugates R_1 and R_0 and is quasiconformal, by group property. This proves Theorem 2.90. □

2.14 On structural stability in the class of quadratic polynomials and Quadratic Fatou Conjecture

Definition 2.95 A polynomial P is *polynomially structurally stable,* if it is structurally stable in the space of polynomials of the same degree $d = degP$.

Let $\Sigma \subset \mathbb{C}$ denote the complement of the closure of the set of those $c \in \mathbb{C}$, for which the corresponding quadratic polynomial $P_c(z) = z^2 + c$ has a parabolic periodic orbit. Let $\Sigma' \subset \Sigma$ denote the complement of the set Σ to the set of those c, for which the orbit of the critical point 0 under the mapping P_c is periodic.

Theorem 2.96 *A quadratic polynomial $P_a(z) = z^2 + a$ is polynomially structurally stable, if and only if $a \in \Sigma'$.*

Theorem 2.97 *(A.Douady, J.Hubbard). The set of those $c \in \mathbb{C}$ for which the polynomial P_c has a parabolic orbit lies in the boundary of the Mandelbrot set.*

We will not give a proof of this theorem.

Corollary 2.98 *The connected components of the set of those $c \in \mathbb{C}$ for which P_c is polynomially structurally stable are*

- the complement to the Mandelbrot set;

- each hyperbolic component of the Mandelbrot set punctured at its center: the parameter c for which the orbit of the point 0 is periodic;

- the queer components (if any).

Consider the mapping

$$\rho : \mathbb{C}^2 \to \mathbb{C}^2; \quad \rho(c, z) = (c, P_c(z)).$$

Lemma 2.99 *Let P_a be polynomially structurally stable. Then there exists a neighborhood $V = V(a)$ and a ρ-invariant holomorphic motion of the whole Riemann sphere over V. Moreover, in the case, when a lies in the interior of the Mandelbrot set, the holonomies H_{ac}, $c \in V$ of the holomorphic motion are conformal conjugacies of the polynomials P_a and P_c on their attracting basins of the infinity.*

Corollary 2.100 *Let $c_1, c_2 \in \mathbb{C}$ lie in the same connected component of the Mandelbrot set. In the case, when this component is hyperbolic, we suppose that c_1 and c_2 are distinct from its center. Then the corresponding polynomials P_{c_1} and P_{c_2} are conjugated by a quasiconformal homeomorphism that is conformal on the attracting basin of infinity and fixes the infinity.*

The lemma implies Theorem 2.96. The proof of the corollary repeats the proof of Theorem 2.90 with obvious changes.

Proof of Lemma 2.99. The proof of the lemma repeats the proof of Theorem 2.89 with the following changes. In the case, when a lies in the interior of the Mandelbrot set, the orbit of the critical point 0 does not tend to infinity. For every $c \in V$ we construct the (unique) conformal conjugacy of germs of the polynomials P_a and P_c at infinity. The family of conjugacies thus constructed depends holomorphically on the parameter c and induces a holomorphic motion of a neighborhood of infinity with conformal holonomies. Taking its pullbacks under the iterates ρ^n extends it to a holomorphic motion of the whole attracting basin of infinity with conformal holonomies, by absence of critical points in the attracting basins of infinity of the polynomials P_c, $c \in V$. The rest of the proof of Theorem 2.89 remain valid in our case without changes. $\square$

Remark 2.101 The quasiconformal mappings send a set of measure zero to a set of measure zero.

Lemma 2.102 *Let c lie in a queer (i.e., non-hyperbolic) component of the Mandelbrot set. Then the Julia set $J = J(P_c)$ has positive area (i.e., Lebesgue measure). Moreover there exists a measurable subset $X \subset J$ of positive measure such that $P_c^{\pm 1}(X) = X$ and there exists a measurable P_c-invariant line field on X.*

Proof The Fatou set $F = F(P_c)$ coincides with the attracting basin of infinity. Indeed, otherwise, if it has a bounded component, then it would have a periodic bounded component, by Sullivan's No Wandering Domain Theorem. The latter would be an attracting basin (being a periodic Fatou component of a structurally stable polynomial). Hence, the basin under question would contain the critical point 0, hence P_c is hyperbolic, – a contradiction. Take an arbitrary other point $s \neq c$ lying in the same queer component. Then P_c is conjugated to P_s by a quasiconformal homeomorphism H, $P_s = H \circ P_c \circ H^{-1}$, and H is conformal on the basin of infinity, that is, on the whole Fatou set. Let $K(z)$ denote the dilatation function of the homeomorphism H (or equivalently, of the H-image of the standard complex structure). It is a P_c-invariant measurable function, since the latter almost complex structure is P_c-invariant and H has L_2 derivatives in the sense of distributions (quasiconformality). Set

$$X = \{z \in \mathbb{C} \mid K(z) \text{ is well-defined, } K(z) > 1 \text{ and } z \notin P_c^n(0) \text{ for every } n \in \mathbb{Z}\}.$$

This is a measurable set such that $P_c^{-1}(X) = X$. One has $X \subset J$, since $K \equiv 0$ on the Fatou set, by conformality.

Case 1): the set X, and hence, J has measure zero. Then H is conformal, since it preserves the standard complex structure as a measurable almost complex structure and by the uniqueness statement of Theorem 1.6. Hence, H is an affine transformation $\mathbb{C} \to \mathbb{C}$ that conjugates P_c and P_s. This may happen only when $c = s$. The contradiction thus obtained shows that this case is impossible.

Case 2): X has positive measure. Let σ denote the image of the standard complex structure under the quasiconformal homeomorphism H. Then for every $z \in X$ σ defines an ellipse in $T_z\mathbb{C}$ that is different from a circle, since $K(z) > 1$. Set l_z to be the line directing its bigger axis. The line field $(l_z)_{z \in X}$ is measurable and P_c-invariant, as is σ. This proves the lemma. □

Conjecture 2.103 (Quadratic No Invariant Line Field Conjecture). Every polynomially structurally stable quadratic polynomial P has no measurable invariant line fields supported on a measurable subset X of positive measure in the Julia set, $P^{\pm 1}(X) = X$.

The Quadratic No Invariant Line Field Conjecture together with the above lemma imply the Quadratic Fatou Conjecture.

2.15 Structural stability, invariant line fields and Teichmüller spaces: general case

Definition 2.104 Let S be a compact orientable surface. Two Riemann surface structures on S are *Teichmüller equivalent,* if they can be transformed one into the other by a quasiconformal homeomorphism $S \to S$ isotopic to identity. Let g denote the genus of the surface S. The space $\mathcal{T}_g$ of Teichmüller equivalence classes is called the *Teichmüller space* (of complex structures on S). This definition extends to punctured surface (S, P), where $P = \{P_1, \dots, P_k\} \subset S$ is a finite subset. Two Riemann surface structures on S are said to be (S, P)-*Teichmüller equivalent*, if they can be transformed one into the other by a quasiconformal homeomorphism of the pair (S, P) (i,e., a quasiconformal homeomorphism preserving P) that is isotopic to identity in the class of homeomorphisms preserving the subset P. The corresponding space of equivalence classes (called *Teichmüller space of punctured surfaces*) will be denoted by $\mathcal{T}_{g,k}$.

Example 2.105 Let S be a torus $\mathbb{T}^2 = S^1 \times S^1$. We mark the standard generators α_1, α_2 of its fundamental group defined by the product structure. Its fundamental group is isomorphic to the first homology group. A homeomorphism of $\mathbb{T}^2$ is isotopic to identity, if and only if it acts trivially on the homology (fundamental group). Its lifting to the universal covering $\mathbb{R}^2$ is a homeomorphism commuting with the action of each generator α_j on $\mathbb{R}^2$. Each complex structure on $\mathbb{T}^2$ makes it a complex torus, the quotient of the line $\mathbb{C}$ by a discrete group $\Gamma =< 1, \mu >$, $\operatorname{Im} \mu > 0$, so that α_1 acts on the universal cover $\mathbb{C}$ by unit translation and α_2 acts by translation by μ. The two last statements together imply that *two complex structures on* $\mathbb{T}^2$ *are Teichmüller equivalent,* if and only if the corresponding numbers μ are equal. Thus, the Teichmüller space $\mathcal{T}_1$ of complex tori is bijectively parametrized by the upper half-plane $\mathbb{H} = \{\operatorname{Im} \mu > 0\} \subset \mathbb{C}$.

Example 2.106 Consider a finitely punctured torus $(\mathbb{T}^2, P)$, $P = \{P_1, \dots, P_k\}$. Then the corresponding Teichmüller space $\mathcal{T}_{1,k}$ has a natural structure of k-dimensional complex manifold. Indeed, there exists a natural projection of the Teichmüller space $\mathcal{T}_{1,k}$ to the space of tuples $(\mu, P_2, \dots, P_k)$, where $\mu \in \mathbb{H}$ is the above modulus of the complex torus,

$$P_2, \dots, P_k \in \mathbb{T}^2_\mu = \mathbb{C}/ < 1, \mu >, \ P_i \neq 0, \ P_i \neq P_j \text{ for } i \neq j.$$

Namely, we normalize the coordinate on the corresponding complex torus by translation to make P_1 equal to 0 and mark the positions of the other points P_j on the complex torus. The above projection, which is a covering map with discrete preimages, equips the Teichmüller space with a structure of k-dimensional complex manifold.

Let R be a structurally stable rational function. To each its attracting n-periodic orbit O we associate a punctured complex torus as follows. The local orbit space of the mapping R^n in a punctured neighborhood of a point $A \in O$ is isomorphic to a complex torus: the orbit space of its linearization $z \mapsto \mu(R)z$ in $\mathbb{C}^*$, here $\mu(R) = (R^n)'(A)$. Let us consider all the critical orbits converging to A. Their projections to the above local orbit space form a finite subset in complex torus; its cardinality equals the number of critical points whose orbits converge to A. Each complex torus has a markedl pair of generators in the homology. Namely, the torus is obtained by glueing a fundamental annulus via the linearized map. One of the generators is given by the generator of the homology of the annulus. The other one is defined by glueing: it is represented by a non-intersected path in the annulus that connects two points glued by the linear map. It is uniquely defined up to addition of a multiple of the first generator. But when R runs a simply connected domain V in the space of structurally stable functions, we can and will choose the latter second generator so that its representative depends continuously on the function $R \in V$. We will denote the corresponding punctured complex torus by $(\mathbb{T}^2_O, P(O))(R)$, $P(O) = \{P_1(O), \dots, P_{k(O)}(O)\}$. It represents a point in the Teichmüller space $\mathcal{T}_{1,k(O)}$. The product

$$\mathcal{T}(R) = \prod_O \mathcal{T}_{1,k(O)}$$

taken over all the attracting periodic orbits will be called here the *Fatou-supported Teichmüller space of the function R.* Recall that each Teichmüller space $\mathcal{T}_{1,k}$ carries a natural structure of complex manifold, whose dimension equals k, if $k \geq 1$.

Remark 2.107 A quasiconformal conjugacy of two rational functions induces a quasiconformal homeomorphism isotopic to identity of the corresponding punctured tori associated to attracting periodic orbits.

Proposition 2.108 *Consider a connected component W of the set of structurally stable rational functions. For every $R_0 \in W$ consider the corresponding attracting periodic orbits $O_1, \dots, O_l$. Fix a simply connected neighborhood $V = V(R_0) \subset W$. Then the orbits under question persist under deformation of the rational function and become functions of $R \in V$ (Implicit Function Theorem.) Consider the mapping $\tau : V \to \mathcal{T}(R_0)$ that sends each function $R \in V$ to the collection of classes of punctured complex tori associated to the corresponding attracting periodic orbits. The mapping τ is locally holomorphic and epimorphic.*

Proof The holomorphicity of the mapping τ follows from construction: the linearizing mappings, the multipliers of the attracting periodic orbits (and hence, the moduli of the corresponding complex tori) and the images of the postcritical points under the linearizing mappings depend holomorphically on the parameters of the rational function. Note that each element of the Teichmüller space $\mathcal{T}(R)$ can be represented by a bounded measurable almost complex structure on the union of the corresponding tori. Every bounded measurable almost complex structure σ on the union of punctured tori associated to a rational function R lifts to a R-invariant almost complex structure $\widetilde{\sigma}$. Let $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the quasiconformal homeomorphism sending $\widetilde{\sigma}$ to the standard complex structure. Then the function $R_\sigma = \Phi \circ R \circ \Phi^{-1}$ is rational, and the corresponding complex tori carry the structure σ. This implies local epimorphicity of the mapping τ and proves the proposition. □

Proposition 2.109 *The Fatou-supported Teichmüller space of a structurally stable rational function R has complex dimension at most $2d - 2$. It equals $2d - 2$, if and only if R is hyperbolic.*

Proof The total number of punctures in complex tori equals the number of the critical points attracted to the attracting periodic orbits. (For structurally stable functions these are exactly the critical points lying in the Fatou set.) Each complex torus has at least one puncture, since each attracting periodic orbit always attracts at least one critical orbit, by Theorem 2.37. Hence, the dimension of the Teichmüller space of each punctured torus equals the number of punctures. Finally, the dimension of the Fatou-supported Teichmüller space equals the number of those critical points, whose orbits converge to attracting orbits. The latter number is no greater that the total number $2d - 2$ of all the critical points and equals $2d - 2$ exactly for hyperbolic functions, by definition. This proves the proposition. □

Corollary 2.110 *Let R_0 be a structurally stable non-hyperbolic rational function. Then there exists a measurable subset $X \subset J$ of positive measure, $R_0^{\pm 1}(X) = X$, such that there exists a measurable R_0-invariant line field on X.*

Proof The space of conformal conjugacy classes of rational functions of degree d is complex $2d-2$-dimensional, since $dim\mathcal{R}^d = 2d + 1$ and $dim PSL_2(\mathbb{C}) = 3$. The value $\tau(R) \in \mathcal{T}(R_0)$ locally depends only on the conformal conjugacy class of the function R. The image of a neighborhood of the function R_0 under the mapping τ has dimension less than $2d-2$, by the above proposition and non-hyperbolicity. Therefore, there exists a rational function R close to R_0 such that $\tau(R) = \tau(R_0)$ that is not conformally conjugated to R_0. The latter equality implies that the conjugating quasiconformal homeomorphism $H = H_R$, $H \circ R_0 \circ H^{-1} = R$ can be chosen conformal on the Fatou set, since the corresponding punctured orbit spaces are conformally equivalent by definition. Let $K(z)$ denote the dilatation function of the homeomorphism H. Set

$$X = \{z \in \mathbb{C} \mid K(z) \text{ is well-defined, } K(z) > 1, \ z \notin R_0^n(c) \text{ for every } n \in \mathbb{Z} \text{ and critical point } c\}.$$

The set X is measurable and has positive measure, and $R_0^{\pm 1}(X) = X$, as in the case of quadratic polynomials, see the proof of Lemma 2.102. Let σ denote the image of the standard complex structure under the quasiconformal homeomorphism H. It is R_0-invariant. The field $(l_z)_{z \in X}$ of lines $l_z \subset T_z\overline{\mathbb{C}}$ directing the bigger axes of the ellipses defined by σ is measurable and R-invariant, as in the proof of Lemma 2.102. This proves the corollary. □

Remark 2.111 It is easy to see that the Lattès examples obtained from integer torus endomorphism (multiplying points of the torus by an integer number) have invariant line fields.

Conjecture 2.112 (No Invariant Line Field Conjecture). A rational map R carries no invariant line field supported on a subset X of positive measure in its Julia set, $R^{\pm 1}(X) = X$, except when R is a Lattès example induced by an integer torus endomorphism.

Conjecture 2.112 together with the above corollary and density of the set of structurally stable rational functions implies the general Fatou Conjecture on density of the set of hyperbolic rational functions.

3 Kleinian groups

In this section we will deal with *discrete and torsion free subgroups* $\Gamma \subset Aut(\overline{\mathbb{C}}) = PSL_2(\mathbb{C})$.

Definition 3.1 The *discontinuity set* is the subset $\Omega = \Omega_\Gamma \subset \overline{\mathbb{C}}$ such that each point $z \in \Omega$ has a neighborhood $U = U(z)$ such that $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \Gamma$. The *limit set* is the complement $\Lambda = \Lambda_\Gamma = \overline{\mathbb{C}} \setminus \Omega$. The group Γ is *non-elementary,* if the cardinality of the set Λ is at least three.

Remark 3.2 Both sets Ω and Λ are Γ-invariant, by definition. For every infinite orbit Γz each its limit point z_0 lies in the limit set, by definition: each neighborhood $U = U(z_0)$ contains an infinite sequence of points $\gamma_n(z)$, $\gamma_n \in \Gamma$, and thus, $\gamma_n^{-1}\gamma_{n+1}(U) \cap U \neq \emptyset$. The quotient Ω/Γ is a Riemann surface, and the projection $\Omega \to \Omega/\Gamma$ is a covering map.

Remark 3.3 Each element $\gamma \in Aut(\overline{\mathbb{C}})$ has one of the three following type:

- *loxodromic,* if it has two fixed points, one attracting and the other one repelling, and the multipliers at both of them are not real;

- *hyperbolic,* if it has two fixed points, one attracting and the other one repelling, and the multipliers at both of them are real;

- *elliptic,* if it is conjugated to a rotation; then it has two fixed points with multipliers of modulus one.

- *parabolic,* it it has one fixed point.

The mappings of the three first types are exactly those mappings that are conformally conjugated to the mappings $z \mapsto \lambda z$ with $\lambda \notin \mathbb{R} \cup \partial D_1$, $\lambda \in \mathbb{R} \setminus \partial D_1$, $|\lambda| = 1$ respectively. The parabolic mappings are those mappings that are conformally conjugated to the translation $z \mapsto z + 1$. The unique fixed point of a translation is the infinity, and it is a parabolic fixed point in the sense of Section 2: in the local coordinate $t = \frac{1}{z}$ the translation takes the form $t \mapsto t + t^2 + \dots$. A discrete torsion free group Γ contains no elliptic elements, since the cyclic group generated by an elliptic element is either finite (and hence, torsion non-free), or non-discrete (if it is a rotation by an angle that is an irrational multiple of π).

Example 3.4 The cyclic group generated by a loxodromic, hyperbolic or parabolic element is elementary. Its discontinuity set is the complement to its fixed points, and Λ is the union of (at most two) fixed points.

Example 3.5 (Schottky group). Consider four disjoint closed disks $\Delta_{ij} \subset \mathbb{C}$, $i, j = 1, 2$, whose centers are vertices of a square with sides parallel to the coordinate x- and y-axes. Namely, we consider that i, j are respectively, the vertical and horizontal coordinates of the center of the disk Δ_{ij}. Consider the group $\Gamma =< a, b >$ generated by the elements a and b acting as follows:

$$a(\overline{\mathbb{C}} \setminus \Delta_{21}) \subset Int(\Delta_{11}); \ b(\overline{\mathbb{C}} \setminus \Delta_{22}) \subset Int(\Delta_{12}).$$

We claim that Γ is the free group with two generators a and b and its limit set is a Cantor set. The freeness follows from the fact that a reduced word of length d in a and b transforms the complement

$$\Pi = \overline{\mathbb{C}} \setminus \cup_{ij} \Delta_{ij}$$

of the four disks inside one of them. Indeed, the last element of the word sends it to one of the disks $\Delta_{i_1 j_1}$. Then the next element sends $\Delta_{i_1 j_1}$ to another disk $\Delta_{i_2 j_2}$ by definition and since the word is reduced: no cancellations aa^{-1}, bb^{-1}. Moreover, the images of the disks Δ_{ij} under a sequence of words of increasing lengths become smaller and smaller and tend to a point after passing to a subsequence. The limit points under question form a Cantor set.

We can do the above construction so that $a(\partial D_{21}) = \partial D_{11}$, $b(\partial D_{22}) = \partial D_{12}$. Then Π is a fundamental domain for the group Γ: its images under the elements of the group Γ are disjoint from it (by the above discussion) and four of them (under action by $a^{\pm 1}$ and $b^{\pm 1}$) are adjacent to it along the boundaries of the disks. This implies that $\overline{\Pi}$ and the union of its images lie in Ω. The same statement implies that the complement to the union of the images of the set Π under action of all the reduced words of length at most d is the union of small disks "of d-th generation", whose radii tend to zero, as $d \to \infty$. The disks of $d+1$-th generation are contained in the disks of d-th generation. This implies that the limit of the union of disks of d-th generation, as $d \to \infty$, is a Cantor set: the one from the above paragraph. This Cantor set contains Λ, since it coincides with the complement of the set $\Gamma\overline{\Pi} \subset \Omega$. But it coincides with Λ: it consists of limit points of Γ-orbit of each point in Π, by construction, and these limit points are contained in Λ, by Remark 3.2. Hence, $\Omega = \Gamma\overline{\Pi}$. The quotient Ω/Γ is a genus two compact Riemann surface obtained by boundary identifications of the domain Π: the element a identifies the left boundary circles, and the element b identify the right ones.

Theorem 3.6 *The limit set Λ is always non-empty and coincides with the closure of the set of fixed points of all the elements of the group Γ.*

Proof The limit set obviously contains the fixed points of its elements, and hence, is non-empty. Let us prove that each point $z_0 \in \Lambda$ is a limit of fixed points.

Claim 1. *There exist two sequences $x_n, y_n \to z_0$ and a sequence of group elements $\gamma_n \in \Gamma$ such that $\gamma_n(x_n) = y_n$.*

Proof For every neighborhood $U = U(z_0)$ there exists a $\gamma \in \Gamma$ such that $\gamma U \cap U \neq \emptyset$, i.e., $\gamma(x) = y$ for some $x, y \in U$, by the definition of limit set. This implies the statement of the claim. $\square$

Claim 2. *In Claim 1 there exists a sequence of fixed points z_n of the elements γ_n that converges to z_0, as $n \to \infty$.*

Proof Passing to a subsequence, without loss of generality we can and will consider that the fixed points A_n, B_n of the elements γ_n converge to some limits A and B. (One has $A_n = B_n$, if γ_n are parabolic.) Suppose the contrary to the claim, i.e., $A, B \neq z_0$. Let us show that γ_n converge to a limit $\gamma \in Aut(\mathbb{C})$ (after passing to a subsequence). This would contradict discreteness of the group Γ and will prove the claim.

Case 1): γ_n are parabolic, $A_n = B_n$ (after passing to a subsequence). Consider a sequence of affine charts on $\overline{\mathbb{C}}$ centered at z_0 so that A_n is the infinity and the charts under question converge to a chart with $A = \infty$. Then the elements γ_n are small translations in the corresponding charts that send a small x_n to a small y_n. Hence, they converge to identity.

Case 2): $A_n \neq B_n$. Consider the cross-ratio function

$$K[x_n, A_n, B_n, z] = \frac{(z - x_n)(B_n - A_n)}{(z - A_n)(B_n - x_n)}.$$

One has

$$K[x_n, A_n, B_n, z] = K[y_n, A_n, B_n, \gamma_n(z)],$$

by invariance of the cross-ratio under conformal automorphisms of the Riemann sphere. The factor $B_n - A_n$ in the expressions of both latter cross-ratios can be cancelled. Taking into account that both $B_n - x_n$ and $B_n - y_n$ tend to $B - z_0 \neq 0$, together with the above equality yields that the Möbius transformations $\frac{z-x_n}{z-A_n}$ and $\frac{\gamma_n(z)-y_n}{\gamma_n(z)-A_n}$ both tend to the Möbius transformation $\frac{z-z_0}{z-A}$, as $n \to \infty$. (Here we use that $A \neq z_0$.) This together with the convergence $y_n \to z_0$ implies that $\gamma_n \to Id$ and proves the claim. $\square$

Claim 2 implies the statement of the theorem. $\square$

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