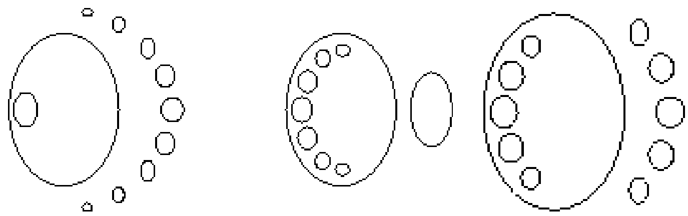
REAL ALGEBRAIC GEOMETRY (1st module)

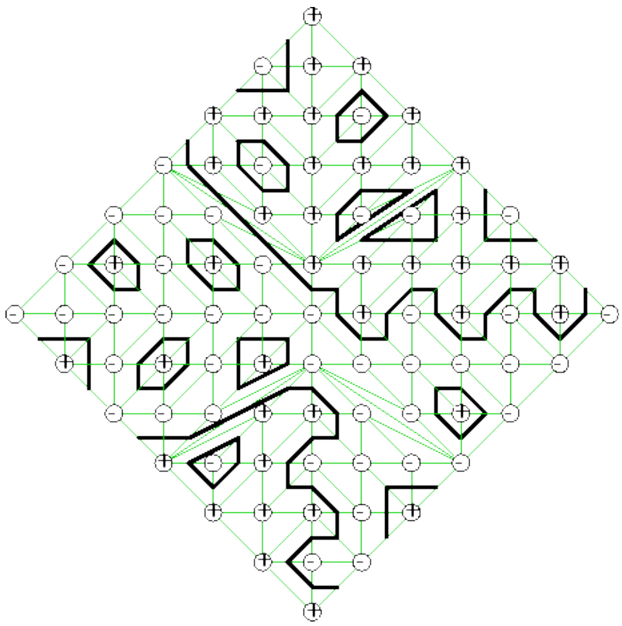
The 16th of the 23 problems for the 20th century, published by David Hilbert in 1900, was devoted to the topology of smooth algebraic curves on the real projective plane. Everybody knows that every such curve of degree 2 is an ellipse, which is the boundary of a topological disc. An interesting problem for a 2nd year student is to prove that every curve of degree 4 consists of the boundaries of ≤4 disjoint discs or two nested discs. For the degree 6, even Hilbert himself was unable to complete the classification: he knew that every such curve consists of the boundaries of ≤11 discs, and, in the case of exactly 11 discs, at most the following three cases are possible:



http://www.pdmi.ras.ru/~olegviro/educ-texts.html

However, he did not know whether the 3rd of these cases takes place for any curve of degree 6. It was not until 1969 that Gudkov constructed an example of such a curve.

Hilbert's 16th problem was the starting point for modern real algebraic geometry (the study of geometry and topology of zero loci of real polynomials), and Gudkov's construction is a special case of one of the most important tools in this science: Viro's patchworking. This tool allows to construct polynomials with a prescribed topological type of the zero locus. For instance, here is the patchworking construction for Gudkov's curve:



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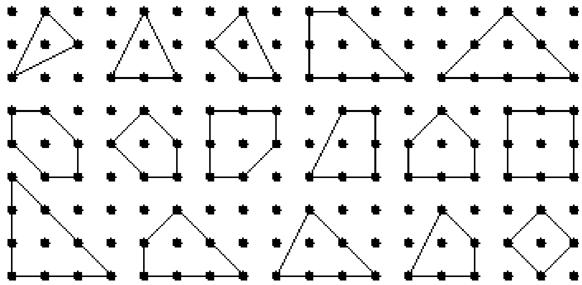
<u>Program</u>: Introduction to real algebraic geometry, Harnack's curve inequality, Viro's patchworking, moment maps, $\mathbb{R}_+$-toric varieties, proof of the patchworking theorem.

<u>References</u>: Viro's textbooks (http://www.pdmi.ras.ru/~olegviro/educ-texts.html)

<u>Prerequisites</u>: The course can be taken by 2nd year students and higher.

TORIC VARIETIES (2nd module)

Every convex polytope with integer coordinates of the vertices gives rise to a certain algebraic variety, which is called toric. This is a remarkable correspondence between varieties and polytopes: it always translates important convex-geometric properties and questions to important algebro-geometric ones, and vice versa. For instance, here is the list of all polygons such that the correpsonding toric surfaces are del Pezzo:



One can readily verify that these are exactly the 16 lattice polygons, containing a unique lattice point in their interior. The platonic dual to any of these polygons is again a polygon on this list, and this convex-geometric symmetry corresponds to the algebro-geometric mirror symmetry for del Pezzo surfaces.

The aim of the course is to study this correspondence between polytopes and varieties and its applications to both convex and algebraic geometry. In particular, applying convex geometry to the study of algebraic geometry, we shall obtain far-reaching generalizations of the classical Bezout theorem (the number of common roots of two polynomials of two variables does not exceed the product of their degrees). In the other direction, applying algebraic geometry to the study of convex geometry, we shall describe relations between the numbers of faces of given dimensions in an arbitrary simplicial polytope.

Program: polytopes, fans and toric varieties; convex-algebro-geometric dictionary; Kouchnirenko theorem; mixed volumes; Bernstein theorem; Dehn–Sommerville equations.

References: Textbooks and works by Kouchnirenko, Bernstein, Khovanskii and Timorin at http://math.hse.ru/nis-12-vgeom

Prerequisites: The course can be taken by 2nd year students and higher.