Introduction to complex dynamics and analytic theory of ordinary differential equations

March 13, 2017

Spring semester, end of January – beginning of May 2018, 2 lectures a week

Complex dynamics and analytic theory of ordinary differential equations are situated on crossing of many domains of contemporary mathematics. The analytic theory of ordinary differential equations and its global extension, the theory of holomorphic foliations were born in the first half of XX-th century, in studying the Painlevé equations and the second part of Hilbert 16-th Problem about limit cycles of *real* planar polynomial vector fields. The Hilbert 16-th Problem asks: is it true that the number of limit cycles is always bounded by a constant depending only on the degrees of the components of the field? It has more than 100-year history. The strongest result, the theorem of finiteness of the number of limit cycles for each individual polynomial vector field was proved by Yu.S.Ilyashenko and simultaneously and independently by J.Ecalle at the end of 1980-ths. Ilyashenko's proof is geometric and based on complex-analytic methods. Studying the 16-th Hilbert Problem led to a lot of important results in local dynamics, normal forms and global properties of holomorphic foliations that became classical.

Now holomorphic foliations and complex dynamics are quickly developing areas on the crossing of several domains in mathematics, including dynamical systems, analysis, complex and riemannian geometry, ergodic theory. For example, both foliations and holomorphic dynamics arise in classification problems in complex geometry and in some problems of mathematical physics.

We plan to start the course by local dynamics, normal forms and moduli of analytic classification. It appears that in many situations the local dynamics is formally conjugated to its formal normal form, but generically, the normalizing series diverges. We will present the geometric explanation of their divergence, the Stokes phenomena providing the complete system of invariants of local analytic classification: Stokes matrices for irregular singular linear equations; Martinet–Ramis moduli for germs of two-dimensional holomorphic vector fields with saddle-node singularity; Ecalle–Voronin moduli for parabolic germs of one-dimensional conformal mappings. We will discuss various applications of the theory to real dynamics and geometry, including a model of Josephson effect in superconductivity and CR geometry.