Affine Kac-Moody Lie algebras

Homework 2

due 18.10.17

1. Prove that any two minimal realizations of a matrix A are isomorphic. Deduce that the algebra $\tilde{L}(A)$ does not depend on a minimal realization.

2. Prove that for any $\lambda \in \mathfrak{h}^*$ the map $\theta_\lambda : \tilde{L}(A) \to \mathrm{End}(T(V))$ defines representation of $\tilde{L}(A)$.

3. Compute the root decomposition for the Lie algebra $\mathfrak{so}_8(\mathbb{C})$.

4. Let $\mathfrak{h}$ be a finite dimensional abelian Lie algebra and V be an $\mathfrak{h}$-module such that $V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha$, where V_α consists of all $v \in V$ such that $hv = \alpha(h)v$ for all $h \in \mathfrak{h}$. Let $U \subset V$ be an $\mathfrak{h}$-submodule. Then $U = \bigoplus_{\alpha\in\mathfrak{h}^*}(U \cap V_\alpha)$.

5. Prove that the Casimir element $ef + fe + h^2/2$ belongs to the center of the universal enveloping algebra $U(\mathfrak{sl}_2)$. Compute the action of the Casimir element on the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$.

6. Describe all one-dimensional central extensions of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.