Hilbert spaces

2.1. Consider the vector space $H = C[-1,1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt$. Let

$$H_0 = \left\{ f \in H : \int_{-1}^{0} f(t)\,dt = \int_{0}^{1} f(t)\,dt \right\}.$$

(a) Prove that H_0 is a closed vector subspace of H.
(b) Does the equality $H = H_0 \oplus H_0^\perp$ hold?

2.2. Prove that every incomplete inner product space H has a closed vector subspace H_0 such that $H_0 \oplus H_0^\perp \neq H$.

2.3. Let $C_c^\infty(a,b)$ be the space of smooth compactly supported functions on the interval (a,b). Prove that for each $p \in [1,\infty)$ $C_c^\infty(a,b)$ is dense in $L^p[a,b]$.

Definition 2.1. Let $f \in L^2[a,b]$. A function $f' \in L^2[a,b]$ is a *weak derivative* of f if

$$\int_a^b f'\varphi\,dt = -\int_a^b f\varphi' dt$$

for all $\varphi \in C_c^\infty(a,b)$.

2.4. Prove that if $f \in L^2[a,b]$ has a weak derivative f', then f' is unique (as an element of $L^2[a,b]$).

2.5 (*the Sobolev space*)**.** Let $W^{1,2}(a,b)$ denote the space of all $f \in L^2[a,b]$ that have a weak derivative $f' \in L^2[a,b]$. Prove that $W^{1,2}(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_a^b (f\bar{g} + f'\bar{g}')\,dt.$$

2.6 (*the Hardy space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let H^2 denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^2\,d\varphi \right)^{1/2} < \infty.$$

Show that the map $f \mapsto (c_n(f))_{n\geqslant 0}$ (where $c_n(f)$ is the nth Taylor coefficient of f at 0) is an isometric isomorphism of $(H^2, \|\cdot\|)$ onto $\ell^2(\mathbb{Z}_{\geqslant 0})$. Hence H^2 is a Hilbert space.

2.7-B (*the Bergman space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $L_a^2(\mathbb{D})$ denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \left(\int_{\mathbb{D}} |f(x+iy)|^2\,dx\,dy \right)^{1/2} < \infty.$$

Show that $L_a^2(\mathbb{D})$ is a closed vector subspace of $L^2(\mathbb{D})$. Hence $L_a^2(\mathbb{D})$ is a Hilbert space.

2.8-B (*the von Neumann–Jordan Theorem*)**.** Let H be a normed space. Suppose that the parallelogram rule holds in H. Show that the formula

$$\langle x, y\rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2 \qquad (x, y \in H)$$

determines an inner product on H, and that the norm generated by $\langle \cdot, \cdot\rangle$ equals the original norm on H.

Linear operators

2.9. Choose $t_0 \in [a,b]$, and consider the linear functional

$$F\colon (C[a,b], \|\cdot\|_p) \to \mathbb{K}, \quad F(x) = x(t_0).$$

(a) Find all $p \in [1,+\infty]$ such that F is bounded. **(b)** Find $\|F\|$.

2.10. Let $\alpha = (\alpha_n) \in \ell^\infty$, and let $X = \ell^p$ or c_0. *The diagonal operator* $M_\alpha\colon X \to X$ takes each $x \in X$ to $(\alpha_n x_n)_{n\in\mathbb{N}} \in X$. **(a)** Show that M_α is bounded. **(b)** Find $\|M_\alpha\|$.

2.11. Let X be either $L^p[0,1]$ $(1 \leqslant p < +\infty)$ or $C[0,1]$. Define $T\colon X \to X$ by

$$(Tf)(x) = \int_0^x f(t)\,dt \qquad (f \in X).$$

(a) Prove that T is bounded. **(b)** Find $\|T\|$ in the cases where $X = C[0,1]$ and $X = L^1[0,1]$.

Remark. If the above operator T acts on $L^2[0,1]$, then $\|T\| = 2/\pi$. We will be able to prove this in due course.

2.12. Let $I = [a,b]$, and let $K \in C(I \times I)$. The *integral operator* $T\colon C(I) \to C(I)$ is given by

$$(Tf)(x) = \int_a^b K(x,y) f(y)\,dy.$$

Prove that T takes $C(I)$ to $C(I)$, that T is bounded, and that $\|T\| \leqslant \|K\|_\infty (b-a)$.

2.13. Let (X,μ) be a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. The *Hilbert-Schmidt integral operator* $T\colon L^2(X,\mu) \to L^2(X,\mu)$ is given by

$$(Tf)(x) = \int_X K(x,y) f(y)\,d\mu(y).$$

Prove that T takes $L^2(X,\mu)$ to $L^2(X,\mu)$, that T is bounded, and that $\|T\| \leqslant \|K\|_2$.

2.14. Define a linear functional F on $(C[0,1], \|\cdot\|_\infty)$ by

$$F(f) = 2f(0) - 3f(1) + \int_0^1 f(t)\,dt.$$

(a) Prove that F is bounded. **(b)** Find $\|F\|$.

2.15. Let X, Y be normed spaces. Suppose that X is finite-dimensional. Prove that each linear operator $T\colon X \to Y$ is bounded.

2.16. Let H^2 be the Hardy space (see Exercise 2.6).
(a) Show that for each $w \in \mathbb{D}$ the linear functional $F_w\colon H^2 \to \mathbb{C}$, $F_w(f) = f'(w)$, is bounded.
(b) The Riesz Theorem together with (a) implies that for each $w \in \mathbb{D}$ there exists a unique $g_w \in H^2$ such that for all $f \in H^2$ we have $F_w(f) = \langle f, g_w\rangle$. Find an explicit formula for $g_w(z)$.

2.17-B. Let $L^2_a(\mathbb{D})$ be the Bergman space (see Exercise 2.7-B).
(a) Show that for each $w \in \mathbb{D}$ the linear functional $F_w\colon L^2_a(\mathbb{D}) \to \mathbb{C}$, $F_w(f) = f(w)$, is bounded.
(b) The Riesz Theorem together with (a) implies that for each $w \in \mathbb{D}$ there exists a unique $g_w \in L^2_a(\mathbb{D})$ such that for all $f \in L^2_a(\mathbb{D})$ we have $F_w(f) = \langle f, g_w\rangle$. Find an explicit formula for $g_w(z)$.