GALOIS THEORY. HW1. (DUE OCTOBER, 6)

1. (The Chinese remainder theorem). Let R be a commutative ring. Ideals $I, J \subset R$ are said to be coprime if $I + J = R$. Show that for any collection $I_1, \cdots I_n$ of pairwise coprime ideals the natural homomorphisms

$$R/I_1 I_2 \cdots I_n \to R/\cap_i I_i \to \bigoplus_i R/I_i$$

are isomorphisms. Here $I_1 I_2 \cdots I_n$ is the product of ideals, that is the ideal generated by all elements of the form $a_1 \cdots a_n$ with $a_i \in I_i$.

2. Let R be a finite dimensional algebra over a field k. Prove that every prime ideal in R is maximal. Also prove that R has only finitely many maximal ideals.

3. Prove that $x^5 - nx - 1 \in \mathbb{Z}[x]$ is irreducible for $n \neq 0, -1, 2$.

4. Prove that, for every prime p and integer $n > 0$, the polynomial

$$f(x) = \frac{x^{p^n} - 1}{x^{p^{n-1}} - 1}$$

is irreducible in $\mathbb{Q}[x]$.

5. Write the real number

$$\frac{1}{1 - 2\sqrt[3]{2}}$$

in the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with $a, b, c \in \mathbb{Q}$.

6. Compute the degrees

(a) $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}]$

(b) $[\mathbb{Q}(\sqrt[3]{2} + \sqrt{2}) : \mathbb{Q}]$

(c) $[\mathbb{Q}(\sqrt{3} + \sqrt{2}) : \mathbb{Q}]$

(d) $[\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) : \mathbb{Q}]$.

(Here, for $\alpha \in \mathbb{R}$, $\mathbb{Q}(\alpha)$ denotes the subfield of $\mathbb{R}$ generated by α.)

7. Prove that, for every prime p and $0 \neq a \in \mathbb{F}_p$, the polynomial $x^p - x - a$ is irreducible in $\mathbb{F}_p[x]$.