Листок 5.

Задача 1. В коробке 7 красных и 5 белых шаров. Случайным образом из коробки вынимают два шара. Найдите математическое ожидание и дисперсию количества красных шаров. Изменится ли ответ, если вынимать шары следующим образом: вытащили первый шар и положили обратно, а затем вытащили второй шар?

Задача 2. Монету, для которой вероятность выпадения «орла» равна p, бросают N раз. Найдите математическое ожидание и дисперсию количества «орлов».

Задача 3. Бросают пять игральных костей. Найдите математическое ожидание и дисперсию суммы выпавших очков.

Задача 4. Сто писем разложили по ста конвертам, на которых уже были написаны адреса, случайным образом. Найдите математическое ожидание количества писем, лежащих в конвертах с правильными адресами.

Задача 5. Монету, для которой вероятность выпадения «орла» равна p, бросают до первого выпадения «орла». Найдите математическое ожидание и дисперсию числа бросаний в случае
(a) число бросаний не более N, причем N-е бросание считается успешным при любом исходе,
(b) число бросаний неограничено.

Задача 6. Случайная величина ξ имеет распределение Пуассона с параметром λ, т. е. для всех целых неотрицательных k верно равенство $P(\xi = k) = \lambda^k e^{-\lambda}/k!$. Найдите математическое ожидание и дисперсию ξ.

Задача 7. Найдите математическое ожидание и дисперсию случайных величин, распределенных следующим образом:

(i) равномерно на отрезке $[-1, 1]$;

(ii) показательно с параметром λ (т.е. плотность распределения имеет вид $\lambda e^{-\lambda t} \mathrm{I}_{t>0}(t)$);

(iii) согласно нормальному закону $\mathcal{N}(m, \sigma^2)$.

Задача 8. Найдите математическое ожидание и дисперсию случайной величины, распределение которой задано плотностью:

(a) $\varrho(x) = 1 - |x|$ при $|x| \le 1$ и $\varrho(x) = 0$ при $|x| > 1$, (b) $\varrho(x) = \frac{1}{2}e^{-|x|}$, (c) $\varrho(x) = \sin x$ при $x \in [0, \pi/2]$ и $\varrho(x) = 0$ при $x \notin [0, \pi/2]$.

Задача 9. Существуют ли независимые непостоянные случайные величины X и Y такие, что $X^2 + Y^2 = 1$.

Задача 10. Опишите все случайные величины ξ такие, что ξ и $\sin \xi$ независимы.

Задача 11. Пусть (x, y) имеет равномерное распределение в (a) $[0, 1]^2$, (b) $\{(x, y) \colon x^2 + y^2 \le 1\}$. Являются ли величины x и y независимыми?

Задача 12. Случайная величина ξ имеет пуассоновское распределение с параметром λ_1, случайная величина η распределена экспоненциально с параметром λ_2, причем ξ и η независимы. Найдите математическое ожидание и дисперсию случайных величин $\xi + \eta, \xi\eta, \max\{\xi, \eta\}$.

Remarks on the problem set 5

Ex 1. Let N be the number of red balls drawn from the box in two extractions. We have $\mathbb{P}(N=1) = \frac{\binom{7}{1}\binom{5}{1}}{\binom{12}{2}} =: p_1$, and $\mathbb{P}(N=2) = \frac{\binom{7}{2}}{\binom{12}{2}} =: p_2$. This entails

$$\mathbb{M}[N] = p_1 * 1 + 2 * p2 = 7/6, \qquad \mathbb{D}[N] = p_1 * 1^2 + p_2 * 2^2 - \mathbb{M}[N]^2 = 175/396.$$

Notice that $N = N_1 + N_2$ where N_i takes the value 1 if the i-th ball was red, and 0 otherwise. Since $\mathbb{P}(N_i = 1) = 7/12 = 1 - \mathbb{P}(N_i = 0)$, from the linearity of the expectation it follows $\mathbb{M}[N] = \mathbb{E}[N_1] + \mathbb{E}[N_2] = 2 * 7/12$.

Now, if we sample with replacement (second question of the exercise), we still have $\mathbb{M}[N] = 7/6$ for the aforementioned reason. However, since now N_1 and N_2 are independent, we gater $\mathbb{D}[N] = \mathbb{D}[N_1] + \mathbb{D}[N_2] = 2 * (7/12)(1 - 7/12) = 35/72$.

Notice that in this second case the variance is higher, and this is quite intuitive. As a limiting case, you can think about sampling 12 balls. If there is no replacement, we will get exactly 7 red balls, thus the variance is 0. If we sample with replacement, then the variance grows with the number of extractions, as we get in this case $12 * (7/12(*1 - 7/12)$.

Ex 2. We can reason as above, in particular in the case of extraction with replacement. Let X =*number of tails (or tails)*. Then $X = \sum_{i=1}^{N} X_i$ where X_i is 1 or 0 depending wether we had a tail at the i-th extraction or not (in particular $\mathbb{M}[X_i] = p$, $\mathbb{D}[X_i] = p(1-p)$). Then $\mathbb{M}[X] = \sum_{i=1}^{N} \mathbb{M}[X_i] = N\,p$. Since the X_i's are independent, $\mathbb{V}[X] = \sum_{i=1}^{N} \mathbb{B}[N_i] = N\,p\,(1-p)$.

Ex 3. Let $n = 6$ be the number of the faces of one die, $k = 5$ the number of tossed dice. If N is the result of one die (say, the first), then we have

$$\mathbb{M}[N] = \sum_{i=1}^{n} \tfrac{1}{n}\, i = \tfrac{n+1}{2} = 7/2$$
$$\mathbb{D}[N] = \sum_{i=1}^{n} \tfrac{1}{6}\, i^2 - \mathbb{M}[N]^2 = \tfrac{n^2-1}{12} = 35/12$$

Since the result of each die is independent from the others, reasoning as in the **Ex 2**, we have that the expectation of the sum of the dice is $k\,\mathbb{M}[N] = 35/2$ and the variance is $k\,\mathbb{D}[N] = 165/12$.

Ex 4. If we select a permutation over n elements (here $n = 100$) randomly (more precisely, giving to each permutation the same probability $1/n!$), the probability that a given number $i \in \{1, \dots, n\}$ is fixed by the permutation is $1/n$. Again, by linearity of the expectation, the expected number of fixed points is $n\,\frac{1}{n} = 1$, regardless of n.

Notice that calculating the variance is also possible in this case (yet not required in the text). As above, let X_i take the value 1 (if i is a fixed point) or 0 (othervise). We have $\mathbb{M}[X_i^2] = \mathbb{M}[X_i] = 1/n$. On the other hand, for $i \neq j$, $\mathbb{M}[X_i\,X_j] = \frac{1}{n(n-1)}$ (since it coincides with the probability that i and j are both fixed). Therefore the variance of the number of fixed points $X = \sum_i X_i$ is calculated as

$$\mathbb{D}[X] = \sum_{i=1}^{n} \mathbb{M}[X_i^2] + \sum_{i \neq j}^{n} \mathbb{M}[X_i\, X_j] \; - \mathbb{M}[X]^2$$
$$n\tfrac{1}{n} + n(n-1)\tfrac{1}{n(n-1)} - 1 = 1$$

We calculated previously (Listok 1, Ex 6) the probability that a permutation over N elements has no fixed point: $\sum_{k=0^N} \frac{(-1)^k}{k!}$. It follows that the probability that a permutation over n elements has exactly ℓ fixed points is

$$p_\ell = \tfrac{1}{n!} \underbrace{\binom{n}{\ell}}_{\text{choose } \ell \text{ fixed points}} \underbrace{(n-\ell)! \sum_{k=1}^{n-\ell} \tfrac{(-1)^k}{k!}}_{\text{perm. with no fixed points on } (n-\ell) \text{ el.}} = \tfrac{1}{\ell!} \sum_{k=1}^{n-\ell} \tfrac{(-1)^k}{k!}$$

We have thus gathered for $n \geq 2$

$$\mathbb{M}[1] = \sum_\ell p_\ell = 1$$
$$\mathbb{M}[X] = \sum_\ell p_\ell\, \ell = 1$$
$$\mathbb{M}[X^2] = \sum_\ell p_\ell\, \ell^2 = 2$$

Can you compute $\mathbb{M}[X^k]$ for $n \geq k$? [Hint: expand the power as we did for the square; Result: The Bell number B_k]

Ex 5. If X is the number of tosses, then we have for (a)

$$p_n := \mathbb{P}_N(X = n) = \begin{cases} (1-p)^{n-1}p & \text{if } 1 \leq n \leq N-1 \\ (1-p)^{N-1} & \text{if } n = N \end{cases}$$

Thus we have

$$\mathbb{M}_N[X] = \sum_{n=1}^{N} np_n = \tfrac{1-(1-p)^N}{p} \mathbb{D}_N[X] = \sum_{n=1}^{N} n^2 p_n - \mathbb{M}[X]^2 = \frac{1 - p - p(1-p)^N(2N-1) - (1-p)^{2N}}{p^2}$$

The case (b) is similar, just we get series at the place of finite sums. In this case $\mathbb{M}[X] = 1/p$ and $\mathbb{D}[X] = (1-p)p^{-2}$.

Ex 6. We have

$$\mathbb{M}[X] = e^{-\lambda} \sum_{k=0}^{\infty} k\lambda^k/k! = e^{-\lambda}\lambda \sum_{k=0} \lambda^k/k! = \lambda$$
$$\mathbb{D}[X] = e^{-\lambda} \sum_{k=0}^{\infty} k^2\lambda^k/k! - M[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Ex 7. We have

(i) $\mathbb{M}[X] = 0$ by symmetry. $\mathbb{D}[X] = \int_{-1}^{1} x^2/2\, dx = 2/3$.
(ii) $\mathbb{M}[X] = \int_0^\infty \lambda t e^{-\lambda t} dt = \lambda^{-1}$ and $\mathbb{D}[X] = 2\lambda^{-2} - \mathbb{M}[X]^2 = \lambda^{-2}$.

(iii) First let's recall that $\int_{\mathbb{R}} \exp(-x^2/2)dx = \sqrt{2\pi}$ (this can be proven by computing the square of such an integral in polar coordinates, or by the means of the residual theorem). Moreover we have (set $y = x^2/2$ and integrate by parts)

$$\int_{\mathbb{R}} x^2 \exp(-x^2/2)dx = \sqrt{2\pi}$$

Therefore the change of variables $z = (x-m)/\sigma$ easily shows that

$$\mathbb{M}[X] = \tfrac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x \exp(-\tfrac{(x-m)^2}{2\sigma^2})dx = m$$
$$\mathbb{D}[X] = \tfrac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} (x-m)^2 \exp(-\tfrac{(x-m)^2}{2\sigma^2})dx = \sigma^2$$

Remark on Ex 9 and Ex 10 Let U be a random variable and $V = f(U)$ for some measurable function f. Then U are V are independent iff V is a.s. constant. Indeed, if V is constant U and V are trivially independent. Conversely, if U and V are independent, then for every measurable set A

$$\begin{aligned} 0 &= \mathbb{P}(V \in \bar{A}, V \in A) = \mathbb{P}(U \in f^{-1}(\bar{A}), V \in A) \\ &= \mathbb{P}(U \in f^{-1}(\bar{A}))\mathbb{P}(V \in A) = \mathbb{P}(V \in \bar{A})\mathbb{P}(V \in A) \end{aligned}$$

Thus for every A the probability of the event $\{V \in A\}$ is either 0 or 1, which easily implies that V is costant a.s..

Notice that the same statement does not hold in the following situation. U is a random variable, and we know that $V = g(U)$ and $W = f(U)$ are independent. In this case we *cannot* deduce that either V or W are a.s. constant. For instance take U uniform in $[-1, 1]$, $V = \text{sign}(U)$ and $W = |V|$.

Ex 9 If X and Y are independent, then X^2 and Y^2 are also independent. And if $X^2 + Y^2 = 1$, then from the previous remark both X^2 and Y^2 are constant a.s., say $X^2 = \cos^2(\theta)$ and $Y^2 = \sin^2(\theta)$ for some angle θ. Thus it must happen $X = U\cos(\theta)$ and $Y = V\sin(\theta)$ where U, V are independent random variables with (a.s.) values in $\{-1, +1\}$.

Ex 10 From the previous remark, it follows that ξ a.s. takes value on the inverse image of sin of some given value $u \in [-1, 1]$.

Ex 11 (a) Yes, since the uniform measure on $[0, 1]^2$ is a product measure (it's enough to check it on intervals). (b) No for instance

$$\mathbb{P}(X > 2^{-1/2}) = \mathbb{P}(Y > 2^{-1/2}) > 0$$
$$\mathbb{P}(X > 2^{-1/2}, Y > 2^{-1/2}) = 0$$