GALOIS THEORY. HW3. (DUE NOVEMBER, 16)

1. Let p be an odd prime number, ϵ a primitive p-th root of 1 in $\mathbb{C}$.
(a) Prove that $\mathbb{Q}(\epsilon)$ contains a unique subfield $K \subset \mathbb{Q}(\epsilon)$ with $[K : \mathbb{Q}] = 2$.
(b) Show that $\sqrt{(-1)^{\frac{p-1}{2}} p} \in \mathbb{Q}(\epsilon)$.

2. Determine the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials
(a) $x^5 + x - 1$
(b) $x^4 - px^2 + q$, where p and q are distinct odd primes.

3. (a) Prove that, for every prime p and a positive integer n, one has the following identy in $\mathbb{F}_p[x]$:

$$x^{p^n} - x = \prod_i f_i(x),$$

where $f_i(x)$ are all irreducible polynomials in $\mathbb{F}_p[x]$ of degrees dividing n.
(b) Find the number of irreducible polynomials in $\mathbb{F}_p[x]$ of degree 5.

4. Let $\mathbb{F}_p(x, y)$ be the field of rational functions in two variables.
(a) Show that the extension $\mathbb{F}_p(x, y) \subset \mathbb{F}_p(x^{\frac{1}{p}}, y^{\frac{1}{p}})$ is not generated by a single element.
(b) Prove that there infitely many fields K with $\mathbb{F}_p(x, y) \subset K \subset \mathbb{F}_p(x^{\frac{1}{p}}, y^{\frac{1}{p}})$.