Topics in 3- and 4-dimensional topology

Below is a list of suggested topics and theorems to speak about. Of course, if a theorem is very deep, then while a proof in the seminar may not be appropriate, there will be a lot to say! Define the terms involved, describe its history, consequences, applications, *sketch* the ideas of the proof or prove a special case, give some examples, etc.

See me for specific references, how to structure your talk or any other guidance. You may work in pairs (or, indeed, you may divide a talk with me).

*****Asterisked***** items need to be covered.

Heegaard Splittings of 3-manifolds

* (Reidemeister-Singer theorem) Any closed, orientable 3-manifold admits a Heegaard splitting, and such a splitting is unique up to isotopy and stabilizations.
• (Waldhausen theorem) Any two Heegaard splittings of the 3-sphere of the same genus are equivalent.
* Define the mapping class group of a surface; prove that it is generated by Dehn twists.
* Classify closed, orientable 3-manifolds Y with Heegaard genus $g(Y) \leq 1$. Show that every non-negative integer can be realized as a Heegaard genus.
* Define Heegaard diagrams; give examples (lens spaces, Seifert manifolds, 3-torus, etc.); express the Reidemeister-Singer theorem in terms of Heegaard diagrams.

(Counter-)examples

• "The eight faces of the Poincaré homology sphere"
`www.maths.ed.ac.uk/ aar/papers/kirbysch.pdf`
• 3-manifolds which have: (a) the same homotopy and co/homology groups, but are not homotopy equivalent; (b) the same homotopy type but are not homeomorphic.

Morse homology

* Define the Morse chain complex, compute some examples. Compare with Floer homology.

Papakyriakopoulos's theorems and consequences

• (Dehn's lemma and the loop theorem) If Y is a 3-manifold and the inclusion homomorphism $\pi_1(\partial Y) \to \pi_1(Y)$ has non-trivial kernel, then there is an embedded 2-disk $D \subset Y$ such that ∂D lies in ∂Y and represents a non-trivial element of $\pi_1(\partial Y)$.
• (The sphere theorem) If Y is an orientable 3-manifold with non-trivial second homotopy group $\pi_2(Y) \neq 0$, then there is an embedded 2-sphere S in Y which is not contractible in Y.

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- (The unknotting theorem) A (smooth) knot $K \subset S^3$ is unknotted if and only if the knot group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.
- (A non-cancelation theorem) If the connect sum $K_1 \# K_2$ of two knots is unknotted, then so are the knots K_1 and K_2.

Surgery on links in 3-space

* (Lickorish-Wallace theorem) Every closed, orientable 3-manifold can be obtained by integral Dehn surgery on a link in the 3-sphere.
* (Kirby theorem) Two closed, orientable 3-manifolds are homeomorphic if and only if they have Kirby diagrams that are related by handle sliding and stabilization (a.k.a. blow up/downs).
* (Rolfsen) Describe Kirby calculus for rational surgeries.

- (Gordon-Luecke theorem) If two knots have homeomorphic complements, then they are equivalent.
- The Property P "Conjecture" (now a theorem)

4-manifolds

- Kirby diagrams for 4-manifolds; the intersection form; Whitehead's theorem: the intersection form determines the homotopy type of simply-connected, closed, oriented 4-manifolds.
- Why is "four" special? That is, survey results that hold in dimensions five and more, but fail in dimension four (and/or below). Describe the Whitney trick, why this fails in dimension four, and the consequences.
- (Trisections of 4-manifolds) A closed, orientable 4-manifold may be decomposed into three 4-dimensional 1-handlebodies of genus g (c.f. a Heegaard splitting of a 3-manifold, a decomposition into two 3-dimensional 1-handlebodies of genus g)

Introducing algebraic geometry: the Casson invariant

* Given a Heegaard splitting $Y = H_1 \cup_\Sigma H_2$ of a homology 3-sphere Y, the Casson invariant counts the <u>algebraic intersection</u> between (conjugacy classes of) <u>irreducible SU_2-representations</u> of the <u>fundamental</u> groups $\pi_1(H_1)$ and $\pi_1(H_2)$. Explain the meaning of the underlined terms.
* Prove independence of choice of Heegaard splitting.
* Define the Casson invariant for knots (via Heegaard splittings) and compute examples.