Linear functionals, dual spaces, dual maps, reflexivity

3.1. Let $X = \mathbb{R}^2$ equipped with the norm $\|\cdot\|_p$, and let $X_0 = \{(x,0) : x \in \mathbb{R}\} \subset X$. Define a linear functional $f_0 \colon X_0 \to \mathbb{R}$ by $f_0(x,0) = x$. We clearly have $\|f_0\| = 1$. Describe all "Hahn-Banach extensions" of f_0, i.e., all linear functionals $f \colon X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $\|f\| = 1$. (Consider all possible $p \in [1, +\infty]$.)

3.2. Does there exist $f \in L^2[0,1]$ such that $\int_0^1 f(t)g(t)\,dt = g(0)$ **(a)** for every polynomial g of degree $\leqslant n$; **(b)** for every polynomial g?

3.3. Let X be a normed space, and let $X_0 \subset X$ be a closed vector subspace. Show that for each $h \in X \setminus X_0$ there exists $f \in X^*$ such that $\|f\| = 1$, $f|_{X_0} = 0$, and $f(h) = \operatorname{dist}(h, X_0)$.

3.4. Construct isometric isomorphisms **(a)** $\ell^\infty \xrightarrow{\sim} (\ell^1)^*$; **(b)** $\ell^1 \xrightarrow{\sim} (c_0)^*$;
(c) $\ell^q \xrightarrow{\sim} (\ell^p)^*$, where $1 < p,q < +\infty$, $1/p + 1/q = 1$. Does this approach give an isometric isomorphism $\ell^1 \cong (\ell^\infty)^*$?

3.5-B. Prove that c_0 is not topologically isomorphic to the dual of a normed space.

3.6-B. Let (X,μ) be a σ-finite measure space. Construct isometric isomorphisms **(a)** $L^\infty(X,\mu) \xrightarrow{\sim} (L^1(X,\mu))^*$; **(b)** $L^p(X,\mu) \xrightarrow{\sim} (L^q(X,\mu))^*$, where $1 < p,q < +\infty$, $1/p + 1/q = 1$.
Hint. To prove the surjectivity of the above maps, apply the Radon-Nikodym theorem.

3.7. Prove that for every infinite-dimensional normed space X there exists an unbounded linear functional on X.

Hint: use the fact that each vector space has an algebraic basis (i.e., a maximal linearly independent set).

3.8. Show that a normed space X is separable iff there exists a dense vector subspace $X_0 \subset X$ of an at most countable dimension.

3.9. Prove that the dimension of an infinite-dimensional Banach space is uncountable.

3.10. Show that c_0, $C[a,b]$, ℓ^p, $L^p[a,b]$, $L^p(\mathbb{R})$ $(p < \infty)$ are separable, while ℓ^∞, $C_b(\mathbb{R})$, $L^\infty[a,b]$, $L^\infty(\mathbb{R})$ are not separable.

3.11. **(a)** Prove that every normed space X can be isometrically embedded into $\ell^\infty(S)$ for some S.
(b) Prove that every separable normed space X can be isometrically embedded into ℓ^∞.

3.12. Let X be a normed space.
(a) Prove that if X^* is separable, then so is X.
(b) Is the converse true?
(c) Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and ℓ^1.

3.13. Prove that
(a) a Hilbert space is reflexive;
(b) c_0 is not reflexive;
(c) ℓ^1 is not reflexive;
(d) $L^1(X,\mu)$ is not reflexive (unless it is finite-dimensional);
(e) $C[a,b]$ is not reflexive.

3.14 (*the dual map*)**.** Let X and Y be normed spaces, and let $T \colon X \to Y$ be a bounded linear map. Define $T^* \colon Y^* \to X^*$ by $T(f) = f \circ T$. Show that T^* is bounded, and that $\|T^*\| = \|T\|$.

3.15. Describe explicitly the duals of the following operators:
(a) the diagonal operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0 (see Exercise 2.10);
(b) the *left shift operator* T_ℓ and the *right shift operator* T_r acting on ℓ^p (where $1 \leqslant p < \infty$) or on c_0 by the rules

$$T_\ell(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \qquad T_r(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots);$$

(c) the *bilateral shift operator* acting on $\ell^p(\mathbb{Z})$ (where $1 \leqslant p < \infty$) or on $c_0(\mathbb{Z})$ by the rule $T(x)_i = x_{i-1}$ ($i \in \mathbb{Z}$);
(d) the "antiderivative" operator on $L^p[0,1]$, $1 \leqslant p < \infty$ (see Exercise 2.11);
(e) the Hilbert-Schmidt integral operator on $L^2(X, \mu)$ (see Exercise 2.13).

3.16. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathscr{B}(X, Y)$ the following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{T} & Y \end{array}$$

3.17. Let X and Y be Banach spaces, and let $T \colon X \to Y$ be a bounded linear map. Show that T is a topological (respectively, isometric) isomorphism iff $T^* \colon Y^* \to X^*$ is a topological (respectively, isometric) isomorphism.

3.18. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Find a relation between the operators $i_{X^*} \colon X^* \to X^{***}$ and $i_X^* \colon X^{***} \to X^*$.

3.19. (a) Prove that a Banach space X is reflexive iff X^* is reflexive.
(b) Deduce that ℓ^1, ℓ^∞, $L^\infty[a,b]$ are not reflexive.