GALOIS THEORY. HW3. (DUE DECEMBER, 13)

1. (a) Let F be a field of characteristic p and let K be a cyclic extension of F of degree p. Prove that $K = F(\alpha)$, where α is a root of the polynomial $x^p - x - a$ for some $a \in F$.
(b) Prove that every field F of positive characteristic such that the algebraic closure $\overline{F}$ is of finite degree over F is algebraically closed.

2. (a) Suppose $F \subset K \subset \overline{F}$ is a finite Galois extension and $F \subset L \subset \overline{F}$ is any finite extension. Then

$$[KL : F] = \frac{[K : F][L : F]}{[K \cap L : F]}.$$

Here $KL \subset \overline{F}$ is the smallest subfield which contains K and L.
(b) Suppose $F \subset K \subset \overline{F}$, $F \subset L \subset \overline{F}$ are finite Galois extensions. Prove that $Gal(KL/F)$ is isomorphic to the subgroup of the direct product $Gal(K/F) \times Gal(L/F)$ consisting of elements whose restrictions to the intersection $K \cap L$ are equal.

3. Prove that the Galois group of $x^6 + 24x - 20$ over $\mathbb{Q}$ is A_6.

4. (a) Given any monic polynomial $P(x) \in \mathbb{Z}[x]$ of degree at least one show that there are infinitely many distinct prime divisors of the integers $P(1), P(2), P(3), P(4), \cdots$. (Hint: Suppose the only primes dividing the values $P(n)$, $n = 1, 2, \cdots$, are the primes $p_1, \cdots p_k$. Let N be an integer with $P(N) = a \neq 0$. Show that $Q(x) = a^{-1}P(N + ap_1 \cdots p_k x)$ is an element of $\mathbb{Z}[x]$ and that $Q(n) \equiv 1 (\mod p_1 \cdots p_k)$, $n = 1, 2, \cdots$.)
(b) Applying (a) to the cyclotomic polynomial show that there are infinitely many primes p with $p \equiv 1 (\mod m)$.