Number Theory. Problem Set II .

0. Prove that every unique factorization Notherian domain of dimension 1 is a principal ideal domain.

1. Prove that every unit in the ring $\mathbb{Z}[\sqrt[3]{2}]$ has the form $\pm(1-\sqrt[3]{2})^k$, $k \in \mathbb{Z}$.

2. Find a system of generators for the group of units in $\mathbb{Z}[\sqrt{19}]$.

3. Find all integral solutions to the equation $3x^2 - 4y^2 = 11$.

4. Let $d > 0$ be an integer. Assume that d is not a complete square. Prove that there is a rational number $\frac{x}{y}$ such that

$$|\sqrt{d} - \frac{x}{y}| < \frac{1}{2y^2}.$$

(Hint: Use the Dirichlet Units Theorem.)

5. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients, $\alpha_i \in \mathbb{C}$, $(1 \leq i \leq n)$, all the roots of $f(x)$. Assume that, for every i, we have

$$|\alpha_i| = 1.$$

Prove that each α_i is a root of 1.

6. Let K be a field of characteristic not equal to 2, $a, b \in K^*$. By definition, the algebra $H_{a,b}(K)$ of generalized quaternions is an algebra over K generated by elements i, j subject ot the relations $ij = -ji$, $i^2 = a$, $j^2 = b$. Set $k := ij$.
(a) Prove that $N(x + yi + zj + tk) = (x^2 - ay^2 - bz^2 + abt^2)^2$.
(b) Prove, that $H_{a,b}(K)$ is either a division algebra or the matrix algebra $Mat_2(K)$.
(c) Prove that $H_{a,b}(K) \xrightarrow{\sim} Mat_2(K)$ if and only if the equation $x^2 - ay^2 - bz^2 = 0$ has a nonzero solution in K.

7. Let $A \subset H_{-1,-1}(\mathbb{Q})$ i and j: $A = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z}$.
(a) Prove that $|Cl(A)| = 2$.
(b) Prove that for every left ideal $0 \neq I \subset A$ the cardinality of A/I is either a complete square or has the form $2n^2$, $n \in \mathbb{Z}$.
Moreover, if $|A/I|$ is a complete square if and only if I is a principal ideal.
(c) Prove that, for every prime $p \neq 2$, one has

$$A/pA \xrightarrow{\sim} Mat_2(\mathbb{F}_p).$$

(d) Prove that for every prime p there exists an ideal $I \subset A$ with $|A/I| = p^2$. How many such ideals are there? (Hint: what are the left ideals $Mat_2(\mathbb{F}_p)$?)
(e) Prove that every positive integer n can be written in the form $x^2 + y^2 + z^2 + t^2$. Moreover, show that the number of such presentations is equal to $8\sum_{d|n, d>0} d$ if n is odd and to $24 \sum_{\substack{m|n \\ m \text{ odd}}} d$ otherwise.

8. Compute the class group of the ring $\mathbb{Z}[\sqrt{-5}]$.

9. What are the prime numbers which can be written in the form $x^2 + 5y^2$ $(x, y \in \mathbb{Z})$? What are the integers which can be written in this form?

10. An order $A \subset D$ in a division algebra over $\mathbb{Q}$ is said to be maximal if D has no strictly larger order. Note , that if A is a maximal order and $0 \neq a \in D$, then aAa^{-1}

is also a maximal order.
(a) Prove that maximal orders exist.
(b) Construct a maximal order in $H_{-1,-1}(\mathbb{Q})$ and compute the class set of this order.
(c)* Show that the number of maximal orders in any division algebra over $\mathbb{Q}$ modulo conjugation is finite. (Hint: show that this number is less or equal to the cardinality of a class set of a maximal order.)