Affine Lie algebras and applications

Exam

due 22.12.2017

1. Prove that for a GCM A one has the following inequalities for the multiplicities $\mathrm{mult}(\alpha) = \dim L(A)_\alpha$:

$$\mathrm{mult}(2(\alpha_i + \alpha_j)) \leq 1,\ \ \mathrm{mult}(\alpha_i + s\alpha_j) \leq 1.$$

2. Let $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. Prove that $\mathrm{mult}(2\alpha_1 + 3\alpha_2) = 2$.

3. Prove that for a 2×2-GCM A one has $\mathrm{mult}(2\alpha_1 + 3\alpha_2) \leq 2$. Find out when $\mathrm{mult}(2\alpha_1 + 3\alpha_2) = 2$.

4. Find all reflections for the action of the affine Weyl group of type $\tilde{A}_2$ (corresponding to the GCM $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$) on the two dimensional Cartan subalgebra of $\mathfrak{sl}_3$. Draw the walls (lines) of these reflections.

5. Find all real roots for the affine Kac-Moody Lie algebra with GCM given by

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

6. Let A be a GCM of finite or affine type. Prove that for a root β and a real root α the string $\{\beta + k\alpha,\ k \in \mathbb{Z}\}$ contains at most five roots. Prove that the length of strings in indefinite type is unbounded.