Distributions

4.1. Show that **(a)** the Dirac δ-function is not regular; **(b)** the derivative of the δ-function is not given by any Radon measure on $\mathbb{R}$.

4.2. **(a)** Show that the rule

$$f(\varphi) = \Big(\mathscr{P}\frac{1}{x}\Big)(\varphi) = \text{v.p.}\int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx$$

determines a distribution f on $\mathbb{R}$.
(b) Show that f is not given by a Radon measure on $\mathbb{R}$.

4.3. Show that **(a)** $\varphi\delta_{x_0} = \varphi(x_0)\delta_{x_0}$ for all tempered $\varphi \in C^\infty(\mathbb{R})$ (where $\delta_{x_0} = \delta(x - x_0)$ is the Dirac δ-function at x_0 given by $\delta_{x_0}(\psi) = \psi(x_0)$); **(b)** $x\mathscr{P}(1/x) = 1$ (where x is the coordinate, i.e., the identity map of $\mathbb{R}$).

4.4. Let $f \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$.
(a) Does $\varphi f = 0$ imply that $f(\varphi) = 0$?
(b) Does $f(\varphi) = 0$ imply that $\varphi f = 0$?

4.5. Let $x_0 \in \mathbb{R}$, and let φ be a smooth tempered function on $\mathbb{R}$. By Exercise 4.3, $\varphi\delta_{x_0} = 0$ iff $\varphi(x_0) = 0$. Find a condition on φ that is necessary and sufficient for $\varphi\delta'_{x_0} = 0$.

4.6. Let φ be a tempered smooth function on $\mathbb{R}$, and let $f \in \mathscr{S}'(\mathbb{R})$. Show that $(\varphi f)' = \varphi' f + \varphi f'$.

4.7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a piecewise C^1-function such that both f and the classical derivative f'_{class} of f (which is defined almost everywhere) are polynomially bounded. Let $\{x_k\}$ be the set of points at which f is discontinuous. Show that $f' = f'_{\text{class}} + \sum_k a_k\delta_{x_k}$, where a_k is the "jump" of f at x_k.

4.8. Construct a continuous and a.e. differentiable function on $\mathbb{R}$ whose classical derivative is not equal to the distributional derivative.

4.9. Calculate the following distributional derivatives: **(a)** $(|x+1| + |x-1|)''$; **(b)** $(\ln|x|)'$; **(c)** $|\sin x|''$.

4.10. Find all $f \in \mathscr{S}'(\mathbb{R})$ satisfying the following equations: **(a)** $f' = 0$; **(b)** $xf = 0$ (where x is the coordinate on $\mathbb{R}$); **(c)** $x^2 f = 0$.

4.11. **(a)** Find the Fourier expansion of $f(x) = x/2 - x^2/4\pi$ on the interval $[0, 2\pi]$ w.r.t. the trigonometric basis e^{inx} ($n \in \mathbb{Z}$).
(b) (the *Poisson summation formula*). Prove the equality $\sum_{n\in\mathbb{Z}} e^{inx} = 2\pi\sum_{n\in\mathbb{Z}} \delta(x - 2\pi n)$ in $\mathscr{S}'(\mathbb{R})$.