

Topics in 3- and 4-dimensional topology

Below is a list of suggested topics and theorems to speak about. Of course, if a theorem is very deep, then while a proof in the seminar may not be appropriate, there will be a lot to say! Define the terms involved, describe its history, consequences, applications, *sketch* the ideas of the proof or prove a special case, give some examples, etc.

See me for specific references, how to structure your talk or any other guidance. You may work in pairs (or, indeed, you may divide a talk with me).

* Asterisked * items need to be covered. † Daggered † items have been previously discussed (but may be reviewed).

Heegaard Splittings of 3-manifolds

- * † (Reidemeister-Singer theorem) Any closed, orientable 3-manifold admits a Heegaard splitting, and such a splitting is unique up to isotopy and stabilizations.
- (Waldhausen theorem) Any two Heegaard splittings of the 3-sphere of the same genus are equivalent.
- * † Define the mapping class group of a surface; prove that it is generated by Dehn twists.
- * † Classify closed, orientable 3-manifolds Y with Heegaard genus $g(Y) \leq 1$. Show that every non-negative integer can be realized as a Heegaard genus.
- * Define Heegaard diagrams; give examples (lens spaces, Seifert manifolds, 3-torus, etc.); express the Reidemeister-Singer theorem in terms of Heegaard diagrams.

(Counter-)examples

- “The eight faces of the Poincaré homology sphere”
www.maths.ed.ac.uk/aar/papers/kirbysch.pdf
- 3-manifolds which have: (a) the same homotopy and co/homology groups, but are not homotopy equivalent; (b) the same homotopy type but are not homeomorphic.

Morse homology

- * Define the Morse chain complex, compute some examples. Compare with Floer homology.

Papakyriakopoulos’s theorems and consequences

- † (Dehn’s lemma and the loop theorem) If Y is a 3-manifold and the inclusion homomorphism $\pi_1(\partial Y) \rightarrow \pi_1(Y)$ has non-trivial kernel, then there is an embedded 2-disk $D \subset Y$ such that ∂D lies in ∂Y and represents a non-trivial element of $\pi_1(\partial Y)$.

† (The sphere theorem) If Y is an orientable 3-manifold with non-trivial second homotopy group $\pi_2(Y) \neq 0$, then there is an embedded 2-sphere S in Y which is not contractible in Y .

† (The unknotting theorem) A (smooth) knot $K \subset S^3$ is unknotted if and only if the knot group $\pi_1(S^3 \setminus K)$ is isomorphic to \mathbb{Z} .

† (A non-cancellation theorem) If the connect sum $K_1 \# K_2$ of two knots is unknotted, then so are the knots K_1 and K_2 .

Surgery on links in 3-space

- * † (Lickorish-Wallace theorem) Every closed, orientable 3-manifold can be obtained by integral Dehn surgery on a link in the 3-sphere.
- * (Kirby theorem) Two closed, orientable 3-manifolds are homeomorphic if and only if they have Kirby diagrams that are related by handle sliding and stabilization (a.k.a. blow up/downs).
- * (Rolfsen) Describe Kirby calculus for rational surgeries.
- (Gordon-Luecke theorem) If two knots have homeomorphic complements, then they are equivalent.

Fibered knots and open book decompositions

- Define fibered knots and links.
- Define the Alexander polynomial $\Delta_K(t)$ of a knot K , and show that it is monic if K is fibered.
- Prove that the trefoil is fibered.
- Every closed, orientable 3-manifold has an open book decomposition.
- Define a contact structure associated to an open book decomposition.

Links of singularities of complex plane curves

- Define the link associated to a complex-valued polynomial with an isolated singularity at the origin, and show that it is fibered.

Introducing algebraic geometry: the Casson invariant

- * Given a Heegaard splitting $Y = H_1 \cup_{\Sigma} H_2$ of a homology 3-sphere Y , the Casson invariant counts the algebraic intersection between (conjugacy classes of) irreducible SU_2 -representations of the fundamental groups $\pi_1(H_1)$ and $\pi_1(H_2)$. Explain the meaning of the underlined terms.
- * Prove independence of choice of Heegaard splitting.
- * Define the Casson invariant for knots (via Heegaard splittings) and compute examples.

Applications of Casson's Invariant

- Define the Rokhlin invariant, and show that it vanishes for any homotopy 3-sphere.
- Discuss the Property P “Conjecture” (now a theorem), and show that if $\Delta_K''(1) \neq 0$, then K has Property P.
- Prove that there exist non-triangulable closed 4-manifolds.

4-manifolds

† Kirby diagrams for 4-manifolds; the intersection form; Whitehead's theorem: the intersection form determines the homotopy type of simply-connected, closed, oriented 4-manifolds.

- Why is “four” special? That is, survey results that hold in dimensions five and more, but fail in dimension four (and/or below). Describe the Whitney trick, why this fails in dimension four, and the consequences.
- (Trisections of 4-manifolds) A closed, orientable 4-manifold may be decomposed into three 4-dimensional 1-handlebodies of genus g (c.f. a Heegaard splitting of a 3-manifold, a decomposition into two 3-dimensional 1-handlebodies of genus g)