Laplace equation. Dirichlet problem

List 5 (26.03.2018)

Deadline — 10-17.04.2017.

1. Write the Laplace operator in $\mathbb{R}^2$ in polar coordinates (r, ϕ).

2. Using the result of the above problem

a) prove that for every $n \in \mathbb{Z}$ a function $f(r)\sin(n\phi)$ is harmonic, if and only if $f(r) = ar^n + br^{-n}$, and the same statement holds with sin replaced by cos;

b) solve the Dirichlet problem $\Delta u = 0$, $u|_{\partial D_1} = u_0$ in the unit disk D_1 with a given C^2-smooth function $u_0 = u_0(\phi)$ on ∂D_1 represented by a Fourier series

$$u_0(\phi) = a_0 + \sum_{k\geq 1}(a_k \sin(k\phi) + b_k \cos(k\phi)).$$

Represent the solution by a Fourier series in ϕ with coefficients depending on r.

3. Solve the Dirichlet problem $\Delta u = 0$, $u|_{\partial D_1} = u_0$ in the unit disk $D_1 \subset \mathbb{R}^2$ with

a) $u_0(\phi) = \sin^2 \phi$,

b) $u_0(\phi) = \cos^3 \phi$,

c) $u_0(\phi) = \sin^3 \phi$.

4. Solve the Dirichlet problem $\Delta u = 0$, $u|_{\partial B_1} = u_0$ in the unit ball $B_1 \subset \mathbb{R}^3$ with $u_0 = \sin(n\phi)$, where $\phi = \phi(x, y, z)$ is the polar coordinate on the (x, y)-plane: it is the longitude identified with the projection of the point $(x, y, z) \in \partial B_1$) to the equator $\{z = 0\} \cap \partial B_1$ along the meridians.

5. Prove that any two C^2-smooth functions f and g on the closure of a piecewise smooth bounded domain $\Omega \subset \mathbb{R}^n$ satisfy the identity

$$\int_{\partial\Omega}(f(\zeta)\frac{\partial g}{\partial \vec{n}}(\zeta) - g(\zeta)\frac{\partial f}{\partial \vec{n}}(\zeta))ds(\zeta) = \int_{\Omega}(f(\zeta)\Delta g(\zeta) - g(\zeta)\Delta f(\zeta))d\zeta_1 \dots d\zeta_n.$$

Here $\vec{n}$ is the field of exterior normal unit vectors on the boundary.

6. Prove the following equivalent definition of the Laplace operator Δ. For every $r > 0$, $x \in \mathbb{R}^n$ and a function u on a neighborhood of the point x containing the closed r-ball centered at x let $S_{r,x}(u)$ denote its average value over the sphere of radius r centered at x. Then

$$\Delta u(x) = \lim_{r\to 0} \frac{2n}{r^2}(S_{r,x}(u) - u(x)).$$

7. Prove that the limit of every uniformly converging sequence of harmonic functions is harmonic. *A possible hint:* use the Poisson formula.

8. Find the derivative in the sense of distributions of the following functions

a) $f(x) = |x|$;

b) $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$;

c) $f(x, y) = r = \sqrt{x^2 + y^2}$.

9. a) Prove that every H_1-Sobolev function of one variable is continuous.

b) Prove that every piecewise smooth function of one variable is H_1-Sobolev.

10. Which ones of the following functions $f(x, y)$ are H_1-Sobolev:

a) the polar coordinate $\phi = \arg \frac{y}{x}$;

b) $\ln r$, where $r = \sqrt{x^2 + y^2}$ is the polar coordinate;

c) $\ln(-\ln r)$?