# Elliptic Functions

## Applications of elliptic integrals

## §2.1 Arithmetic-geometric mean

Well-known fact: geometric mean $\leqq$ arithmetic mean:

(G<A) $$\sqrt{ab} \leqq \frac{a+b}{2}$$

for $a, b > 0$.

Repeat the processs: $a_0 = a$, $b_0 := b$ for $a \geqq b > 0$,

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}.$$

Because of the inequality (G<A),

$$b_n \leqq b_{n+1} \leqq a_{n+1} \leqq a_n,$$

i.e.,

$$b = b_0 \leqq b_1 \leqq \cdots \leqq b_n \leqq \cdots \leqq a_n \leqq \cdots \leqq a_1 \leqq a_0 = a.$$

$\{a_n\}_{n=0,1,2,\ldots}$, $\{b_n\}_{n=0,1,2,\ldots}$: bounded monotone sequences.

$\Longrightarrow \exists\alpha := \lim_{n\to\infty} a_n,\ \beta := \lim_{n\to\infty} b_n$.

In fact, $\alpha = \beta$, i.e., $c_n := \dfrac{1}{2}(a_n - b_n) \to 0$.

<u>Proof</u>:

$$0 \leqq a_n - b_n \leqq a_n - b_{n-1} = \frac{a_{n-1} + b_{n-1}}{2} - b_{n-1} = c_{n-1}.$$

$$\Longrightarrow\ 0 \leq c_n \leqq \frac{c_{n-1}}{2} \leqq \cdots \leqq \frac{c_0}{2^n} \to 0 \qquad (n \to \infty). \quad \square$$

$M(a, b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ : *arithmetic-geometric mean (AGM).*

Easy to see:

(i) $M(a,b) = M(b,a)$.

(ii) $M(\lambda a, \lambda b) = \lambda M(a,b)$.

(iii) $M(a,a) = a$.

(iv) $M(a,b) = M(a_n, b_n)$ $(n = 0, 1, 2, \dots)$.

(ii) $\Longrightarrow$ enough to study $M(a,1)$ or $M(1,b)$ for $a > 1 > b$.

<u>Theorem</u>: If $0 < k < 1$, then $M(1,k) = \frac{\pi}{2} \frac{1}{K(k')}$, where

- $k' := \sqrt{1-k^2}$,
- $K(k')$: complete elliptic integral of the first kind,

$$K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2\theta}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2 z^2)}}.$$

Proof:

Lemma: for $a \geqq b > 0$,

$$I(a,b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \frac{\pi}{2}\frac{1}{M(a,b)}.$$

Once this lemma is proved, the theorem is its corollary:

$$M(1,k) = \frac{\pi}{2}\frac{1}{I(1,k)},$$

and

$$\begin{aligned} I(1,k) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + k^2\sin^2\theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}} = K(k'). \end{aligned}$$

□

Proof of the lemma:

Key:

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

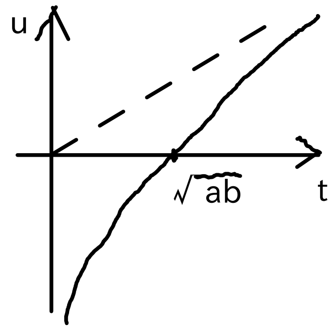
In fact,

$$\begin{aligned} I(a,b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\cos\theta\sqrt{a^2 + b^2\tan^2\theta}} \\ &= \int_0^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}, \qquad \left(\begin{array}{c} t = b\tan\theta,\ \dfrac{dt}{d\theta} = \dfrac{b}{\cos^2\theta}, \\ b^2 + t^2 = \dfrac{b^2}{\cos^2\theta} \end{array}\right). \end{aligned}$$

Another change of the integration variable: $u = \dfrac{1}{2}\left(t - \dfrac{ab}{t}\right)$,

$\dfrac{du}{dt} = \dfrac{1}{2}\left(1 + \dfrac{ab}{t^2}\right) = \dfrac{\sqrt{ab+u^2}}{t}$, $(a^2+t^2)(b^2+t^2) = t^2((a+b)^2+4u^2)$.

(Figure: graph of $u = u(t)$.)




$$I(a,b) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{((a+b)^2+4u^2)(ab+u^2)}}$$
$$= 2\int_0^{\infty} \frac{du}{\sqrt{((a+b)^2+4u^2)(ab+u^2)}}$$
$$= \int_0^{\infty} \frac{du}{\sqrt{\left(\left(\frac{a+b}{2}\right)^2+u^2\right)((\sqrt{ab})^2+u^2)}}$$

Comparing the expression of $I(a,b)$ in the last two pages, we have

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

Repeating the procedure,

$$I(a,b) = I(a_1,b_1) = \cdots = I(a_n,b_n) = \cdots \longrightarrow I\big(M(a,b), M(a,b)\big).$$

In general,

$$\begin{aligned} I(c,c) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{c^2\cos\theta + c^2\sin^2\theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{c\sqrt{\cos\theta + \sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{c} = \frac{\pi}{2c}. \end{aligned}$$

$$\Longrightarrow I(a,b) = \frac{\pi}{2M(a,b)}. \qquad \square$$

Similarly, when $k > 1$,

$$M(k,1) = \frac{\pi}{2}\frac{1}{K(k')}, \qquad k' := \sqrt{1-k^2}.$$

<u>Remark</u>: $k' \notin \mathbb{R}$ but $K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}$ makes sense.

<u>Example</u>: $k = \sqrt{2}$ ($k' = \sqrt{-1}$): $K(\sqrt{-1}) = \frac{\pi}{2M(\sqrt{2},1)}$, i.e.,

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2M(\sqrt{2},1)}.$$

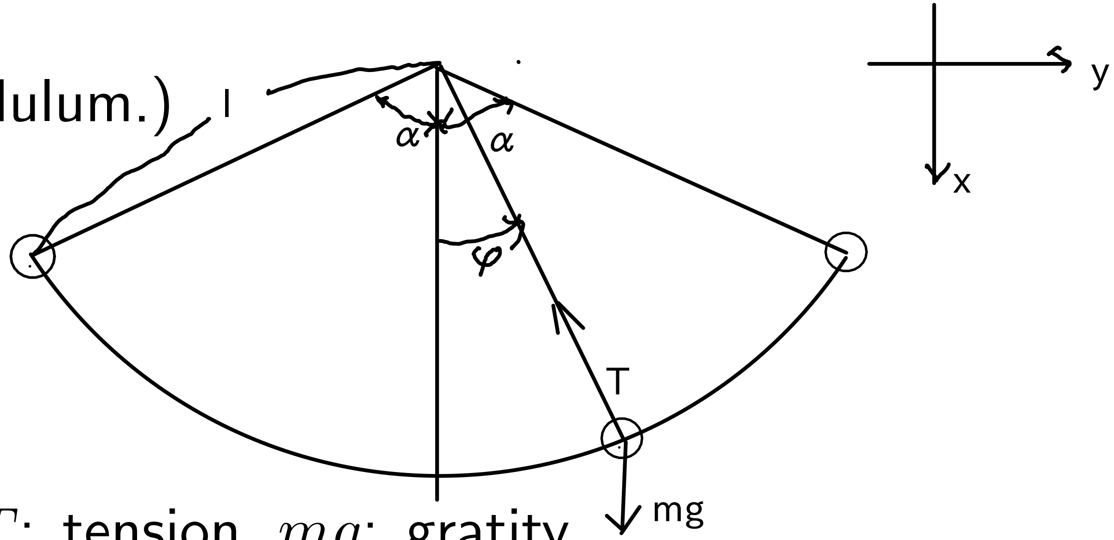
Gauß found this formula by computing the both sides *by hand*!

From $a_4$ and $b_4$: $M(\sqrt{2},1) = 1.981402347... = \frac{\pi}{2\int_0^1 \frac{dx}{\sqrt{1-x^4}}}$.

(cf. Gauß, Gesammelte Werke, Tagebuch.)

# §2.2 Motion of a simple pendulum

(Figure of a simple pendulum.)




$l$: length of the string, $T$: tension, $mg$: gratity.

$(x(t), y(t)) = (l\cos\varphi(t), l\sin\varphi(t))$: coordinates of the point mass.

$$\begin{aligned}\text{acceleration} &= \frac{d^2}{dt^2}(l\cos\varphi(t), l\sin\varphi(t)) \\ &= \frac{d}{dt}(-l\dot{\varphi}\,\sin\varphi, l\dot{\varphi}\,\cos\varphi) \\ &= (-l\ddot{\varphi}\,\sin\varphi - l\dot{\varphi}^2\,\cos\varphi, l\ddot{\varphi}\,\cos\varphi - l\dot{\varphi}^2\,\sin\varphi)\end{aligned}$$

The equation of motion:

$$ml \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \ddot{\varphi} - ml \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} \dot{\varphi}^2 = \begin{pmatrix} -T\cos\varphi + mg \\ -T\sin\varphi \end{pmatrix}.$$

$x$-component $\times(-\sin\varphi)+$ $y$-component $\times\cos\varphi$: $ml\ddot{\varphi} = -mg\sin\varphi$, i.e.,

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \sin\varphi, \qquad \omega := \sqrt{\frac{g}{l}}.$$

Linear approximation: amplitude $\ll 1 \Longrightarrow \sin\varphi \approx \varphi$.

$$\frac{d^2\varphi}{dt^2} = -\omega^2\varphi.$$

A *LINEAR* differential equation of the second order.

General solution: $\varphi(t) = c_1\cos\omega t + c_2\sin\omega t$, $c_1, c_2 \in \mathbb{R}$.

Exact solutions without approximation $\sin\varphi \approx \varphi$.

(eq. of motion) $\times\dot{\varphi}$:

$$\begin{aligned}
\frac{d^2\varphi}{dt^2}\frac{d\varphi}{dt} &= -\omega^2 \sin\varphi \frac{d\varphi}{dt}, \\
\frac{1}{2}\frac{d}{dt}\left(\frac{d\varphi}{dt}\right)^2 &= \frac{d}{dt}(\omega^2\cos\varphi) \\
\xrightarrow{\int(\cdots)dt} \qquad \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 &= \omega^2\cos\varphi + (\text{const.}),
\end{aligned}$$

$$\tilde{E} := \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 - \omega^2\cos\varphi = (\text{const.}),$$

$\approx$ conservation law of the energy.

Remark: the total energy of the pendulum is

$$(\text{kinetic energy}) + (\text{potential energy}) = \frac{ml^2}{2}\left(\frac{d\varphi}{dt}\right)^2 - mgl\cos\varphi = ml^2\tilde{E}.$$

$\alpha :=$ maximum amplitude, i.e., if $\varphi(t_0) = \alpha$, $\dot{\varphi}(t_0) = 0$.

$\Longrightarrow \tilde{E} = -\omega^2 \cos\alpha$, hence

$$\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 = \omega^2(\cos\varphi - \cos\alpha) = 2\omega^2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}\right),$$

$$\frac{d\varphi}{dt} = 2\omega\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}}.$$

Note: $\left|\sin\frac{\varphi}{2}\right| < \sin\frac{\alpha}{2}$, because $|\varphi| \leqq \alpha < \pi$.

Notations: $k := \sin\frac{\alpha}{2}$, $\theta := \arcsin\left(k^{-1}\sin\frac{\varphi}{2}\right)$, i.e., $\sin\frac{\varphi}{2} = k\sin\theta$.

The equation of motion becomes

$$\frac{d\varphi}{dt} = 2\omega\sqrt{k^2 - k^2\sin^2\theta} = 2k\omega\cos\theta.$$

On the other hand, $\dfrac{d\varphi}{dt} = \dfrac{d\varphi}{d\theta}\dfrac{d\theta}{dt}$ and

$$\frac{d}{d\theta} k\sin\theta = \frac{d}{d\theta}\left(\sin\frac{\varphi}{2}\right), \text{ i.e., } k\cos\theta = \frac{\cos\frac{\varphi}{2}}{2}\frac{d\varphi}{d\theta} = \frac{\sqrt{1-k^2\sin^2\theta}}{2}\frac{d\varphi}{d\theta}$$

Hence,

$$\begin{aligned}
\frac{2k\cos\theta}{\sqrt{1-k^2\sin^2\theta}}\frac{d\theta}{dt} &= 2k\omega\cos\theta, \\
\frac{1}{\sqrt{1-k^2\sin^2\theta}}\frac{d\theta}{dt} &= \omega, \\
\int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} &= \int_0^t \omega\, dt = \omega t.
\end{aligned}$$

(Normalisation: $\theta(0) = 0$, i.e., $\varphi(0) = 0$.)

The left hand side $= F(k, \theta(t)) =$ elliptic integral of the first kind.

The motion of the pendulum:

$$t(\theta) = \frac{1}{\omega} F(k, \theta) = \sqrt{\frac{l}{g}} F\left(\sin\frac{\alpha}{2}, \theta\right).$$

$$\text{Period} = 4(\text{time between } \varphi = 0 \text{ and } \varphi = \alpha)$$

$$= 4(\text{time between } \theta = 0 \text{ and } \theta = \frac{\pi}{2}) = 4\sqrt{\frac{l}{g}} K\left(\sin\frac{\alpha}{2}\right).$$

(The period depends on the amplitude!)

Since $K\left(\sin\frac{\alpha}{2}\right) = \frac{\pi}{2}\frac{1}{M(1, \cos\frac{\alpha}{2})}$,

$$\text{Period}(\alpha) = \frac{\text{Period}(\alpha = 0)}{M(1, \cos\frac{\alpha}{2})}.$$

The pendulum knows the AGM!