

Elliptic Functions

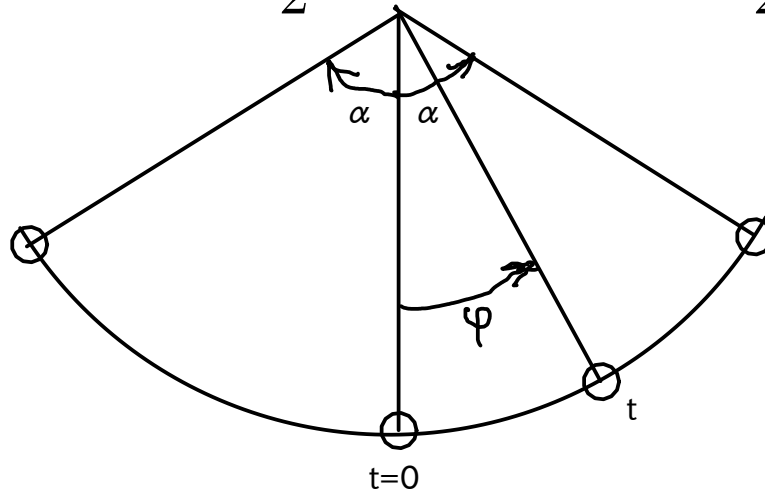
Jacobi's elliptic functions over \mathbb{R}

§3.1 Jacobi's elliptic functions

Recall: the motion of a simple pendulum is describe by

$$t(\theta) = \sqrt{\frac{l}{g}} F\left(\sin \frac{\alpha}{2}, \theta\right) = \sqrt{\frac{l}{g}} \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where $\alpha = \text{max. amplitude}$, $k = \sin \frac{\alpha}{2}$, $\sin \theta = k^{-1} \sin \frac{\varphi}{2}$.



Better description of motion: “amplitude $\varphi = \text{function of time } t$ ”.

\implies Consider the *inverse function*!

Assume $0 \leq k < 1$.

Definition:

Jacobi's elliptic function $\operatorname{sn}(u) = \operatorname{sn}(u, k) :=$ the inverse function of

$$u(x) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

on $-1 \leq x \leq 1$ and, consequently,

$-K(k) \leq u \leq K(k)$ (= the complete elliptic integral of the first kind.)

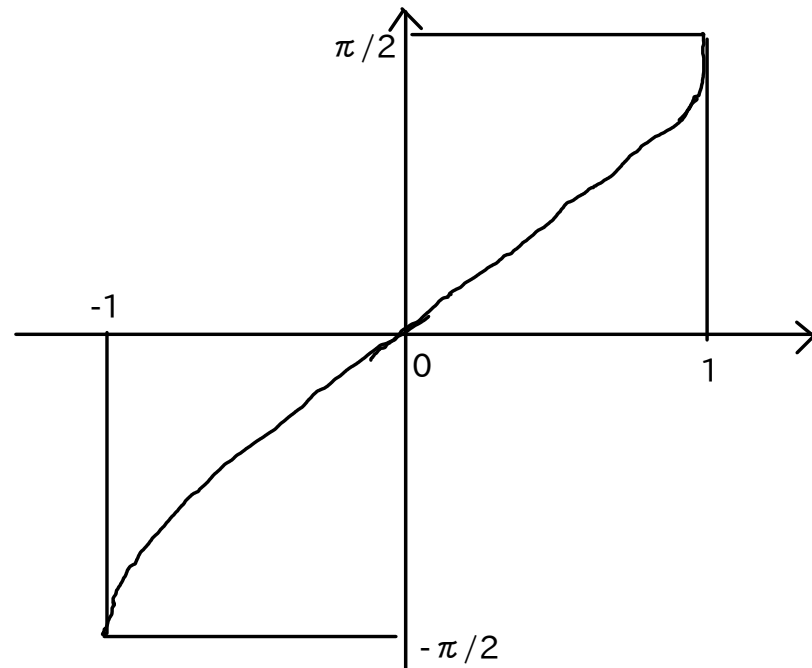
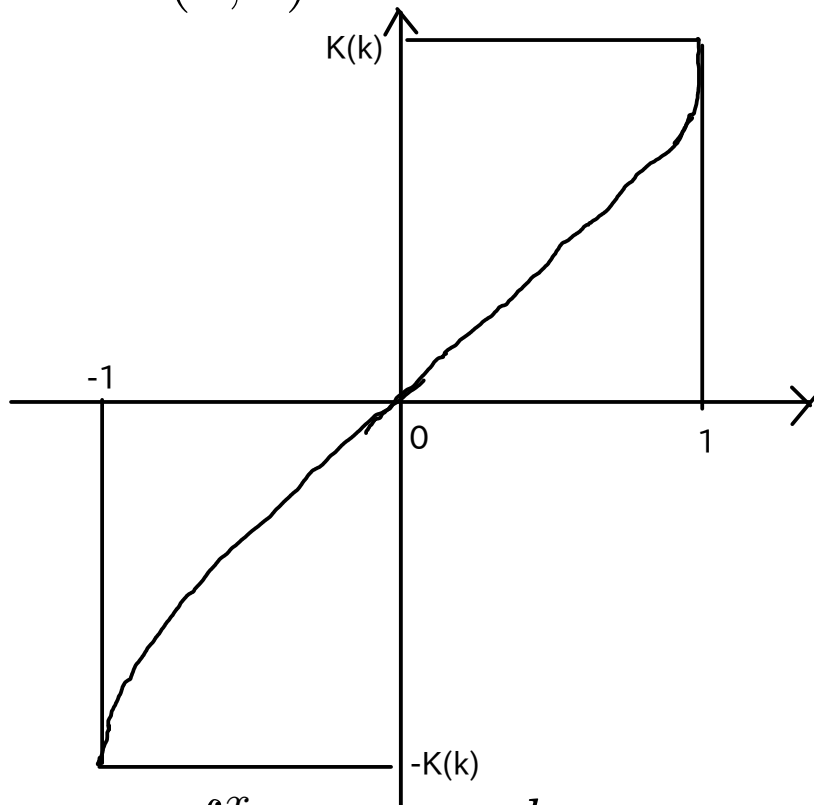
$\operatorname{sn}(u, k) =$ analogue of $\sin u$: in fact, when $k = 0$,

$$\operatorname{sn}(u, k) = \text{the inverse function of } \left(u(x) = \int_0^x \frac{dz}{\sqrt{(1-z^2)}} = \arcsin x \right).$$

Namely,

$$\operatorname{sn}(u, 0) = \sin u, \quad K(0) = \frac{\pi}{2}.$$

Graphs of $F(x, k)$ and $\arcsin x$:



$$F(x, k) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \xrightarrow{k \rightarrow 0} \int_0^x \frac{dz}{\sqrt{1-z^2}} = \arcsin x$$

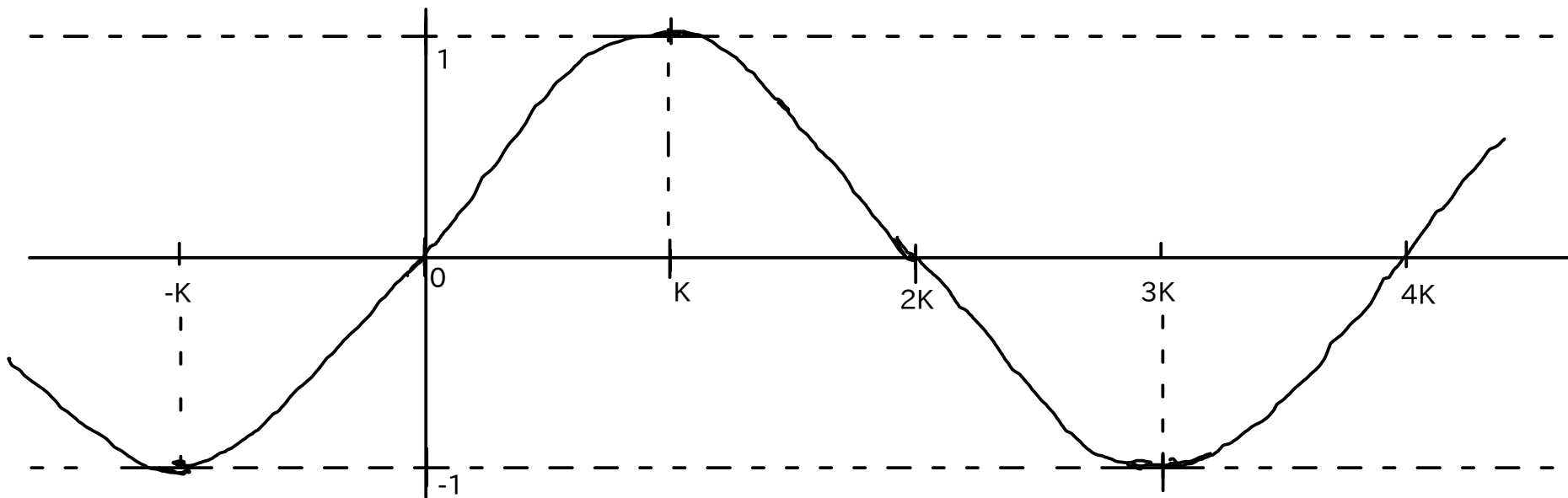
Important property of \sin = periodicity: $\sin(u + 2\pi) = \sin u$.

\implies Extend sn to \mathbb{R} by periodicity:

$$\operatorname{sn}(u + 2K(k), k) = -\operatorname{sn}(u, k), \quad \operatorname{sn}(u + 4K(k), k) = \operatorname{sn}(u, k).$$

(Justification given in § “Complex elliptic integrals/functions”.)

Graph of sn



Introduce cn (analogue of \cos) and dn :

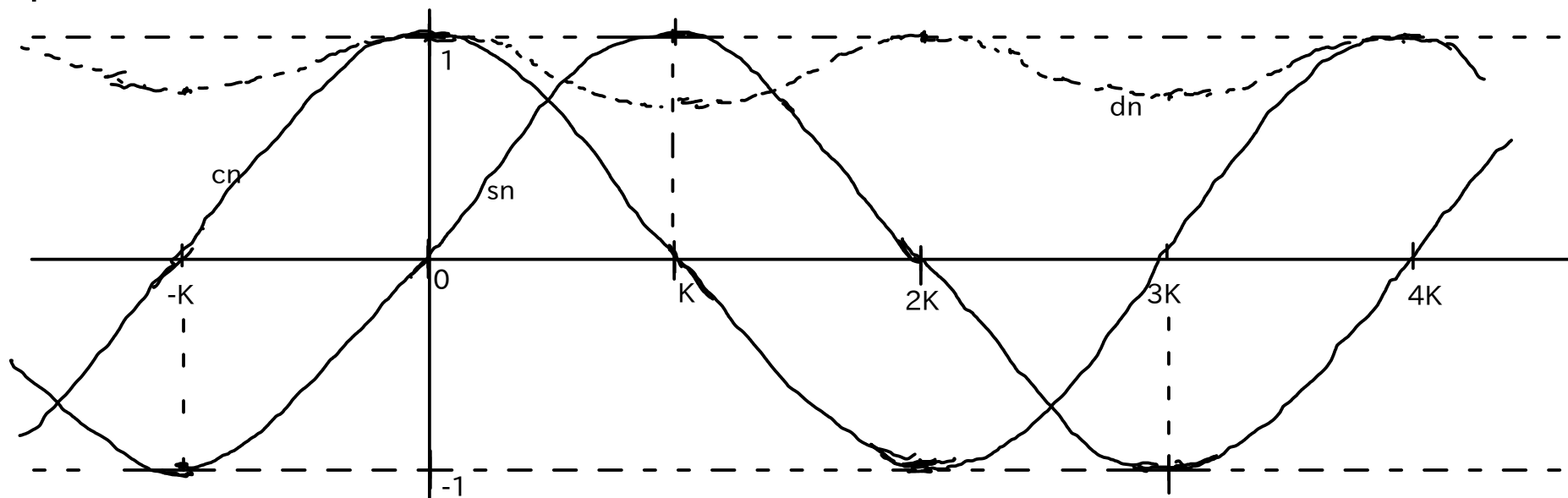
$$\text{cn}(u) = \text{cn}(u, k) := \sqrt{1 - \text{sn}^2(u, k)}, \quad (\text{cn}(0) = 1),$$

$$\text{dn}(u) = \text{dn}(u, k) := \sqrt{1 - k^2 \text{sn}^2(u, k)}, \quad (\text{dn}(0) = 1),$$

and extend by periodicity.

$k \rightarrow 0$: $K(k) \rightarrow \pi$, $\text{sn } u \rightarrow \sin u$, $\text{cn } u \rightarrow \cos u$, $\text{dn } u \rightarrow 1$.

Graphs of sn , cn and dn :



Exercise: Show that, when $k \rightarrow 1$,

$$K(k) \rightarrow \infty,$$

$$\operatorname{sn}(u, k) \rightarrow \tanh u = \frac{\sinh u}{\cosh u},$$

$$\operatorname{cn}(u, k), \operatorname{dn}(u, k) \rightarrow \operatorname{sech} u = \frac{1}{\cosh u}.$$

$\operatorname{sn} u =$ the inverse function of elliptic integral $F(k, x)$, i.e.,

$$u = F(k, \operatorname{sn} u) = \int_0^{\operatorname{sn} u} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

$$\frac{d}{du} \operatorname{sn} u = \frac{1}{\left. \frac{\partial}{\partial x} F(k, x) \right|_{x=\operatorname{sn} u}} = \frac{1}{\left. \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \right|_{x=\operatorname{sn} u}}$$

$$= \sqrt{1 - \operatorname{sn}^2 u} \sqrt{1 - k^2 \operatorname{sn}^2 u} = \operatorname{cn} u \operatorname{dn} u.$$

As corollaries,

$$\frac{d}{du} \operatorname{cn} u = \frac{d}{du} \sqrt{1 - \operatorname{sn}^2 u} = \frac{-\operatorname{sn} u \frac{d \operatorname{sn} u}{du}}{\sqrt{1 - \operatorname{sn}^2 u}} = -\operatorname{sn} u \operatorname{dn} u.$$

$$\frac{d}{du} \operatorname{dn} u = \frac{d}{du} \sqrt{1 - k^2 \operatorname{sn}^2 u} = \frac{-k^2 \operatorname{sn} u \frac{d \operatorname{sn} u}{du}}{\sqrt{1 - k^2 \operatorname{sn}^2 u}} = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

Summarising,

$$\frac{d \operatorname{sn} u}{du} = \operatorname{cn} u \operatorname{dn} u,$$

$$\frac{d \operatorname{cn} u}{du} = -\operatorname{sn} u \operatorname{dn} u,$$

$$\frac{d \operatorname{dn} u}{du} = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

$$\xrightarrow{k \rightarrow 0}$$

$$\frac{d \sin u}{du} = \cos u,$$

$$\frac{d \cos u}{du} = -\sin u,$$

$$\xrightarrow{k \rightarrow 0}$$

Addition formulae:

Addition formula of sin: $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

Addition formula of tanh: $\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$.

$\operatorname{sn}(u, k)$ interpolates sin ($k = 0$) and tanh ($k = 1$).

\implies A natural guess is “sn has an addition formula.”

Let us “interpolate” the above formulae!

Addition formula of sin without cos:

$$\sin(u + v) = \sin u \frac{d \sin v}{dv} + \frac{d \sin u}{du} \sin v.$$

Note $\frac{d \tanh u}{du} = 1 - \tanh^2 u$. Hence

$$\tanh u \frac{d \tanh v}{dv} + \frac{d \tanh u}{du} \tanh v = (\tanh u + \tanh v)(1 - \tanh u \tanh v).$$

Addition formula of tanh can be rewritten as

$$\tanh(u + v) = \frac{\tanh u \frac{d \tanh v}{dv} + \frac{d \tanh u}{du} \tanh v}{1 - \tanh^2 u \tanh^2 v}.$$

A possible interpolation of the addition formulae of sin and tanh:

$$\begin{aligned} \operatorname{sn}(u + v) &= \frac{\operatorname{sn} u \frac{d \operatorname{sn} v}{dv} + \frac{d \operatorname{sn} u}{du} \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \\ &= \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \end{aligned}$$

In fact this is true!

Proof: $u + v \rightarrow c, v \rightarrow c - u,$

$$F(u) := \frac{\operatorname{sn} u \operatorname{cn}(c - u) \operatorname{dn}(c - u) + \operatorname{sn}(c - u) \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2(c - u)}.$$

Claim: $\frac{dF}{du} = 0$, when c is fixed.

Claim $\implies F(0) = F(u)$ and, since $\operatorname{sn} 0 = 0, \operatorname{cn} 0 = \operatorname{dn} 0 = 1,$

$$\operatorname{sn} c = \frac{\operatorname{sn} u \operatorname{cn}(c - u) \operatorname{dn}(c - u) + \operatorname{sn}(c - u) \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2(c - u)}.$$

Substituting $c = u + v$, we obtain the addition formula. □

Proof of the claim:

$N :=$ numerator of $F(u) = \operatorname{sn} u \operatorname{cn}(c - u) \operatorname{dn}(c - u) + \operatorname{sn}(c - u) \operatorname{cn} u \operatorname{dn} u$

$D :=$ denominator of $F(u) = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2(c - u).$

Long computation (Exercise!) shows

$$\frac{dN}{du} D = N \frac{dD}{du}.$$

Therefore

$$\frac{dF}{du} = \frac{\frac{dN}{du} D - N \frac{dD}{du}}{D^2} = 0.$$

□

Addition formulae of cn and dn:

$$\begin{aligned} \operatorname{cn}(u+v) &= \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ \operatorname{dn}(u+v) &= \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \end{aligned}$$

§3.2 Jacobi's function in physics

The motion of the simple pendulum revisited:

$$\begin{aligned}t(\theta) &= \sqrt{\frac{l}{g}} \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad \left(k = \sin \frac{\alpha}{2}\right) \\ &= \sqrt{\frac{l}{g}} \int_0^{\sin \theta} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} \quad (z = \sin \phi) \\ &= \sqrt{\frac{l}{g}} \int_0^{k^{-1} \sin \frac{\varphi}{2}} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.\end{aligned}$$

Using Jacobi's sn function,

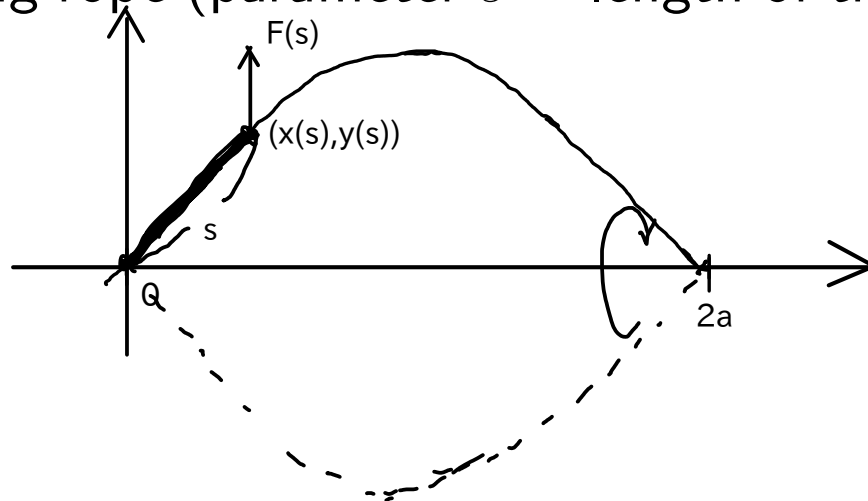
$$\sin \frac{\varphi(t)}{2} = k \operatorname{sn} \left(\sqrt{\frac{g}{l}} t, k \right), \quad \varphi(t) = 2 \arcsin \left(k \operatorname{sn} \left(\sqrt{\frac{g}{l}} t, k \right) \right).$$

Another application: form of a skipping rope

Assumptions:

- The rope rotates fast enough. \implies centrifugal force \gg gravity.
- The density ρ (= mass/length) of the rope is constant.
- The ends are fixed at $(0, 0)$ and $(2a, 0)$ in the xy -plane.
- The rope rotates around the x -axis with the angular velocity ω .

Figure of the skipping rope (parameter s = length of the rope from $(0, 0)$):



The centrifugal force at $(x, y) = (\text{mass}) \times \omega^2 y$.

Potential at (x, y) :

$$V(y) := -\frac{\text{mass}}{2} \omega^2 y^2, \text{ i.e., } -\frac{\partial V(x, y)}{\partial y} = \text{centrifugal force.}$$

mass of the segment of length $ds = \rho ds$.

$$\text{Total potential: } U = -\frac{1}{2} \int_0^l \omega^2 y(s)^2 \rho ds = -\frac{\rho \omega^2}{2} \int_0^l y(s)^2 ds.$$

When the form of the skipping rope is stable, U is minimum.

Constraint: the length of the rope $= l$.

The problem to be solved: a variational problem for $y = y(x)$:

- Maximise $\int_0^l y(s)^2 ds = \int_0^{2a} y(x)^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.
- Under the constraint $l = \int_0^l ds = \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

Calculus of variations with the Lagrange multiplier:

$$\begin{aligned}\mathcal{L}[y(x)] &:= \int_0^{2a} y(x)^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx - \lambda \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{2a} L[y, y'] dx, \quad L[y, y'] := (y^2 - \lambda) \sqrt{1 + y'^2}.\end{aligned}$$

Want: $y_0(x)$, such that

$$\left. \frac{\delta \mathcal{L}}{\delta y} \right|_{y_0} = 0, \quad y_0(0) = y_0(2a) = 0, \quad l = \int_0^{2a} \sqrt{1 + y_0'^2} dx.$$

Variation $\delta \mathcal{L} = \mathcal{L}[y + \delta y] - \mathcal{L}[y]$

$$\begin{aligned} &= \int_0^{2a} (L[y + \delta y, y' + \delta y'] - L[y, y']) dx \\ &= \int_0^{2a} \left(\frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial y'} \delta y' + o(\delta y, \delta y') \right) dx \\ &= \int_0^{2a} \left(\frac{\partial L}{\partial y} \delta y - \frac{d}{dx} \frac{\partial L}{\partial y'} \delta y + o(\delta y, \delta y') \right) dx \quad (\text{integration by parts}) \\ &= \int_0^{2a} \left(\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right) \delta y dx + o(\delta y, \delta y') \end{aligned}$$

$\delta\mathcal{L}$ should be $o(\delta y, \delta y')$ for any variation $\delta y(x)$

$$\implies \text{Euler-Lagrange equation: } \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0.$$

In our case ($L = (y^2 - \lambda)\sqrt{1 + y'^2}$),

$$\frac{\partial L}{\partial y} = 2y\sqrt{1 + y'^2},$$

$$\begin{aligned} \frac{d}{dx} \frac{\partial L}{\partial y'} &= \frac{d}{dx} \left((y^2 - \lambda) \frac{y'}{\sqrt{1 + y'^2}} \right) \\ &= 2y \frac{y'^2}{\sqrt{1 + y'^2}} + (y^2 - \lambda) \left(\frac{y''}{\sqrt{1 + y'^2}} - \frac{y'^2 y''}{(1 + y'^2)^{3/2}} \right). \end{aligned}$$

Euler-Lagrange equation $\times \sqrt{1 + y'^2}$:

$$2y - (y^2 - \lambda) \frac{y''}{1 + y'^2} = 0, \quad \text{i.e., } \frac{y''}{1 + y'^2} = \frac{2y}{y^2 - \lambda}.$$

Note: from “physical” point of view, $y'' < 0$ in the region $\{x > 0, y > 0\}$.
Hence $\lambda > y^2$.

Integral of the Euler-Lagrange equation:

$$\frac{1}{2} \log(1 + y'^2) = \log(\lambda - y^2) + (\text{const.}),$$

$$\text{therefore,} \quad 1 + y'^2 = C(\lambda - y^2)^2.$$

Let $x = b$ be the maximum value of $y(x)$:

$$b = y(x_0), \text{ where } y'(x_0) = 0.$$

$$\implies C = (\lambda - b^2)^{-2}.$$

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= \left(\frac{\lambda - y^2}{\lambda - b^2}\right)^2 - 1 \\ &= \frac{(\lambda - y^2)^2 - (\lambda - b^2)^2}{(\lambda - b^2)^2} = \frac{(b^2 - y^2)(2\lambda - b^2 - y^2)}{(\lambda - b^2)^2}. \end{aligned}$$

$\eta := y/b$ satisfies

$$\frac{d\eta}{dx} = c\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}, \text{ where } c^2 = \frac{b^2(2\lambda - b^2)}{(\lambda - b^2)^2}, \quad k^2 = \frac{b^2}{2\lambda - b^2}.$$

Integrate by x :

$$\int \frac{d\eta}{\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}} = \int c dx.$$

The integral of the LHS = elliptic integral of the first kind!

$$\eta = \operatorname{sn}(cx), \text{ i.e., } y = b \operatorname{sn}(cx).$$

b, c : to be determined by the condition “length of the rope = l ”.