

Elliptic Functions

Elliptic curves

§5.1 Riemann surfaces of $\sqrt{\varphi(x)}$, $\deg \varphi = 3, 4$

Want: elliptic integrals $\int R(x, \sqrt{\varphi(x)}) dx$ with complex variables.

\implies Need: the Riemann surface \mathcal{R} of $\sqrt{\varphi(x)}$, $\deg \varphi = 3, 4$.

The construction is the same as the case of \sqrt{z} , $\sqrt{1-z^2}$.

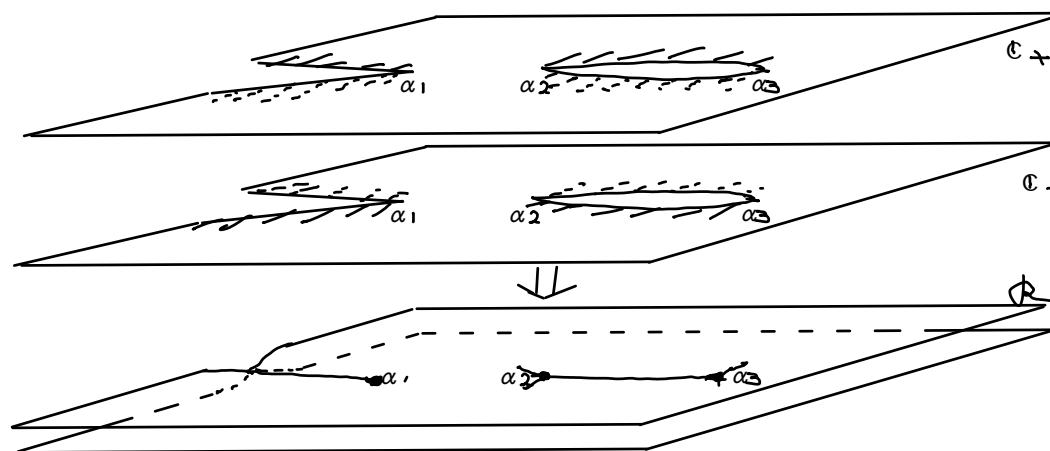
- $\deg \varphi(x) = 3.$

$$\varphi(z) = a(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$$

$a \neq 0, \alpha_1, \alpha_2, \alpha_3$: distinct.

The Riemann surface \mathcal{R} of $\sqrt{\varphi(x)}$

= two copies of $\mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\}$ glued $\cup \{\alpha_1, \alpha_2, \alpha_3\}$.



$$\begin{aligned} \mathcal{R} &= (\mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\})_+ \cup \{\alpha_1, \alpha_2, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_1, \alpha_2, \alpha_3\})_- \\ &= \{(z, w) \mid w^2 = \varphi(z)\}. \end{aligned}$$

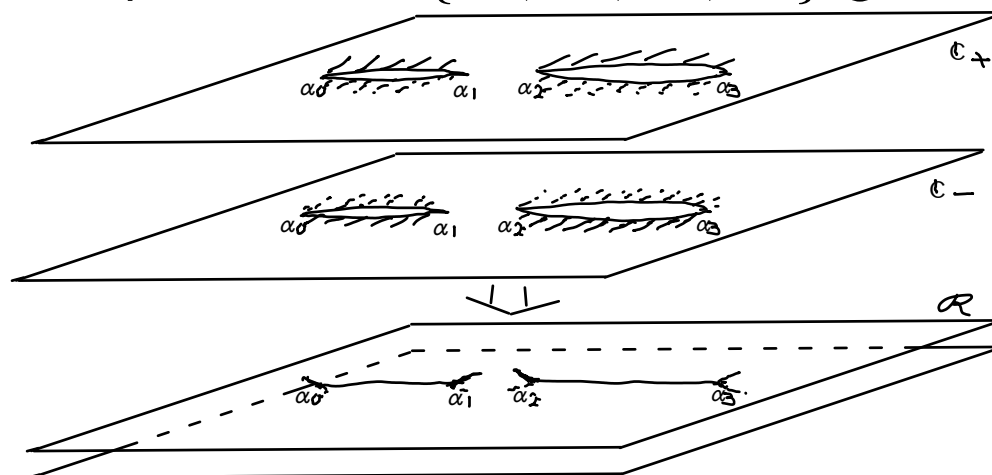
- $\deg \varphi(x) = 4.$

$$\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$$

$a \neq 0, \alpha_0, \alpha_1, \alpha_2, \alpha_3$: distinct.

The Riemann surface \mathcal{R} of $\sqrt{\varphi(x)}$

= two copies of $\mathbb{C} \setminus \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ glued $\cup \{\alpha_1, \alpha_2, \alpha_3\}$.



$$\begin{aligned} \mathcal{R} &= (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_+ \cup \{\alpha_0, \dots, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_- \\ &= \{(z, w) \mid w^2 = \varphi(z)\}. \end{aligned}$$

Proposition:

For both cases, $\deg \varphi(z) = 3$ and 4 ,

(i) $\mathcal{R} = \{(z, w) \mid w^2 = \varphi(z)\}$: a non-singular algebraic curve.

($\iff \left(F, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial w}\right) \neq (0, 0, 0)$, where $F(z, w) = w^2 - \varphi(z)$).

(ii) $\sqrt{\varphi(z)} = w$: holomorphic on \mathcal{R} .

(iii) 1-form $\omega = \frac{dz}{\sqrt{\varphi(z)}} = \frac{dz}{w}$: holomorphic on \mathcal{R} .

Exercise: Check these statements.

§5.2 Compactification and elliptic curves

When $\deg \varphi(z) > 2$, $\int_{z_0}^{\infty} \frac{dz}{\sqrt{\varphi(z)}}$ converges.

\implies Need to add ∞ to \mathcal{R} (*Compactification*).

• $\deg \varphi = 3$.

Use the embedding into the *projective plane* \mathbb{P}^2 :

$$\begin{aligned} \mathcal{R} = \{(z, w) \mid w^2 = \varphi(z)\} \subset \mathbb{C}^2 &\hookrightarrow \mathbb{P}^2 \\ (z, w) &\mapsto [1 : z : w]. \end{aligned}$$

Recall:

$$\begin{aligned} \mathbb{P}^2 &= \mathbb{C}^3 \setminus \{0\} / \sim, \\ (a, b, c) \sim (a', b', c') &\iff \exists \lambda \neq 0, (\lambda a, \lambda b, \lambda c) = (a', b', c'). \end{aligned}$$

Embedding of \mathbb{C}^2 into \mathbb{P}^2 :

$$\mathbb{C}^2 \ni (z, w) \mapsto [1 : z : w] \in \mathbb{P}^2,$$

$$\mathbb{P}^2 \supset U_0 := \{[x_0 : x_1 : x_2] \mid x_0 \neq 0\} \ni [x_0 : x_1 : x_2] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \mathbb{C}^2.$$

\mathcal{R} as a subset of $U_0 \subset \mathbb{P}^2$: $\left(\frac{x_2}{x_0}\right)^2 = \varphi\left(\frac{x_1}{x_0}\right)$, i.e.,

$$x_0 x_2^2 = a(x_1 - \alpha_1 x_0)(x_1 - \alpha_2 x_0)(x_1 - \alpha_3 x_0). \quad (*)$$

Extend \mathcal{R} by this equation:

$$\bar{\mathcal{R}} := \{[x_0 : x_1 : x_2] \mid (*)\} \subset \mathbb{P}^2.$$

What points are added to \mathcal{R} ?

Since $\mathbb{P}^2 \setminus U_0 = \{[x_0 : x_1 : x_2] \mid x_0 = 0\}$,

$$\begin{aligned}\bar{\mathcal{R}} \setminus \mathcal{R} &= \{[x_0 : x_1 : x_2] \mid x_0 = 0, (*)\} \\ &= \{[x_0 : x_1 : x_2] \mid x_0 = 0 = ax_1^3\} \\ &= \{[x_0 : x_1 : x_2] \mid x_0 = x_1 = 0\} = \{[0 : 0 : 1]\}\end{aligned}$$

Namely, $\bar{\mathcal{R}} = \mathcal{R} \cup \{\infty\}$, $\infty = [0 : 0 : 1]$.

The coordinates of \mathbb{P}^2 in the neighbourhood of ∞ : $(\xi, \eta) := \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$.

$$\begin{aligned} (*) &\iff \frac{x_0}{x_2} = a \left(\frac{x_1}{x_2} - \alpha_1 \frac{x_0}{x_2}\right) \left(\frac{x_1}{x_2} - \alpha_2 \frac{x_0}{x_2}\right) \left(\frac{x_1}{x_2} - \alpha_3 \frac{x_0}{x_2}\right) \\ &\iff \xi = a(\eta - \alpha_1\xi)(\eta - \alpha_2\xi)(\eta - \alpha_3\xi).\end{aligned}$$

Exercise:

Check that the equation

$$\xi = a(\eta - \alpha_1\xi)(\eta - \alpha_2\xi)(\eta - \alpha_3\xi)$$

defines a non-singular algebraic curve in the nbd of $(\xi, \eta) = (0, 0)$.

$\bar{\mathcal{R}}$ = defined by equation (*) \implies closed in \mathbb{P}^2 } $\implies \bar{\mathcal{R}}$: compact.
 \mathbb{P}^2 : compact

$\implies \bar{\mathcal{R}}$ is a compact Riemann surface, a *compactification* of \mathcal{R} .

• $\deg \varphi = 4$.

$$\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3).$$

Want: compactification of $\mathcal{R} = \{(z, w) \mid w^2 - \varphi(z) = 0\}$.

Try the same procedure as before:

$$\begin{aligned} \mathcal{R} \subset \mathbb{C}^2 &\hookrightarrow \mathbb{P}^2 \\ (z, w) &\mapsto [1 : z : w]. \end{aligned}$$

The homogeneous equation for \mathcal{R} :

$$\begin{aligned} \left(\frac{x_2}{x_0}\right)^2 - a \left(\frac{x_1}{x_0} - \alpha_0\right) \left(\frac{x_1}{x_0} - \alpha_1\right) \left(\frac{x_1}{x_0} - \alpha_2\right) \left(\frac{x_1}{x_0} - \alpha_3\right) &= 0, \\ \text{i.e., } x_0^2 x_2^2 - a(x_1 - \alpha_0 x_0) \cdots (x_1 - \alpha_3 x_0) &= 0. \quad ((**)) \end{aligned}$$

As before $\{[x_0 : x_1 : x_2] \mid (**)\} = \mathcal{R} \cup \{\infty = [0 : 0 : 1]\}$.

Alas! $\infty = [0 : 0 : 1]$ is a singular point!

Exercise: Check this.

Another compactification:

Instead of \mathbb{P}^2 , use $X := W \cup W' / \sim$, where

$$W = \mathbb{C}^2 \ni (z, w), \quad W' = \mathbb{C}^2 \ni (\xi, \eta),$$

$$(z, w) \sim (\xi, \eta) \iff z\xi = 1, \quad w = \frac{\eta}{\xi^2}$$

$\mathcal{R} \subset W$ as before. \implies the equation of $\mathcal{R} \cap W'$:

$$\left(\frac{\eta}{\xi^2}\right)^2 - a \left(\frac{1}{\xi} - \alpha_0\right) \left(\frac{1}{\xi} - \alpha_1\right) \left(\frac{1}{\xi} - \alpha_2\right) \left(\frac{1}{\xi} - \alpha_3\right) = 0,$$

$$\text{i.e., } \eta^2 - a(1 - \alpha_0\xi)(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi) = 0.$$

$$\mathcal{R}' := \{(\xi, \eta) \in W' \mid \eta^2 = a(1 - \alpha_0\xi)(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi)\}.$$

a non-singular algebraic curve as before.

$$\bar{\mathcal{R}} := \mathcal{R} \cup \mathcal{R}' \subset X = W \cup W'.$$

What is $\bar{\mathcal{R}}$?

What point lies in $\bar{\mathcal{R}} \setminus \mathcal{R}$?

$$W' \setminus W = \{(\xi = 0, \eta) \mid \eta \in \mathbb{C}\}$$

$$\implies \mathcal{R}' \setminus \mathcal{R} = \{(0, \eta) \mid \eta^2 = a\} = \{(0, \pm\sqrt{a})\} \subset W'.$$

They do not belong to W , i.e., they are “infinities”: $\infty_{\pm} := (0, \pm\sqrt{a})_{W'}$.

$$\bar{\mathcal{R}} = \mathcal{R} \cup \{\infty_+, \infty_-\}.$$

Interpretation of $\bar{\mathcal{R}}$ by the gluing construction:

$$\begin{aligned}\mathcal{R} &= (\text{the Riemann surface of } w = \sqrt{\varphi(z)}) \\ &= (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_+ \cup \{\alpha_0, \dots, \alpha_3\} \cup (\mathbb{C} \setminus \{\alpha_0, \dots, \alpha_3\})_-.\end{aligned}$$

- When $\alpha_i \neq 0$ for $\forall i = 0, \dots, 3$.

Denote $\beta_i := \alpha_i^{-1}$.

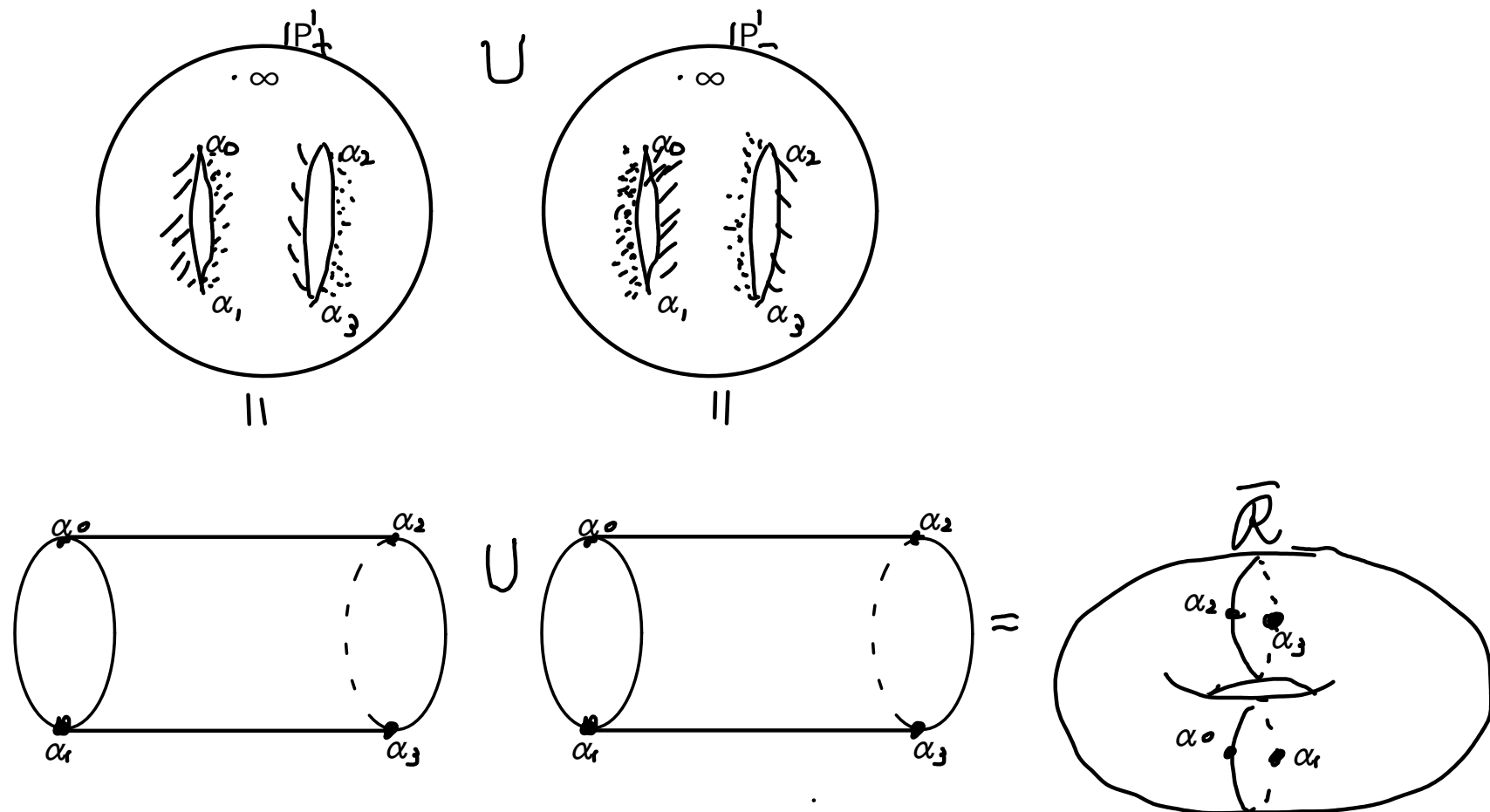
$$\begin{aligned}\mathcal{R}' &= (\text{the Riemann surface of } \eta = \sqrt{a(1 - \alpha_0\xi) \cdots (1 - \alpha_3\xi)}) \\ &= (\mathbb{C} \setminus \{\beta_0, \dots, \beta_3\})_+ \cup \{\beta_0, \dots, \beta_3\} \cup (\mathbb{C} \setminus \{\beta_0, \dots, \beta_3\})_-.\end{aligned}$$

$$\begin{cases} \xi_+ \longleftrightarrow z_+ = \frac{1}{\xi_+} \\ 0_+ \longleftrightarrow \infty_+ \end{cases}, \quad \beta_i \longleftrightarrow \alpha_i = \frac{1}{\beta_i}, \quad \begin{cases} \xi_- \longleftrightarrow z_- = \frac{1}{\xi_-} \\ 0_- \longleftrightarrow \infty_- \end{cases},$$

$\implies \bar{\mathcal{R}}$ is constructed by gluing two $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$'s together.

$$\bar{\mathcal{R}} = (\mathbb{P}^1 \setminus \{\alpha_0, \dots, \alpha_3\})_+ \cup \{\alpha_0, \dots, \alpha_3\} \cup (\mathbb{P}^1 \setminus \{\alpha_0, \dots, \alpha_3\})_-.$$

Figure: two \mathbb{P}^1 's with cuts $\alpha_0\alpha_1$ & $\alpha_2\alpha_3$ glued together \cong a torus:



- When one of α_i 's (say α_0) = 0.

$$\begin{aligned}\mathcal{R}' &= (\text{the Riemann surface of } \eta = \sqrt{a(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi)}) \\ &= (\mathbb{C} \setminus \{\beta_1, \beta_2, \beta_3\})_+ \cup \{\beta_1, \beta_2, \beta_3\} \cup (\mathbb{C} \setminus \{\beta_1, \beta_2, \beta_3\})_-.\end{aligned}$$

\implies everything is the same as before.

Definition: *Elliptic curve*: compactification of $\{(z, w) \in \mathbb{C}^2 \mid w^2 = \varphi(z)\}$,
 $\deg \varphi(z) = 3$ or 4 .

Remark: When $\deg \varphi \geq 5$: *hyperelliptic curve*.

Recall: Elliptic integrals are reduced to

$$\int R(x, \sqrt{(1-x^2)(1-k^2x^2)}) dx$$

by means of fractional linear transformations.

The same is true for elliptic curves:

Any elliptic curves are isomorphic to

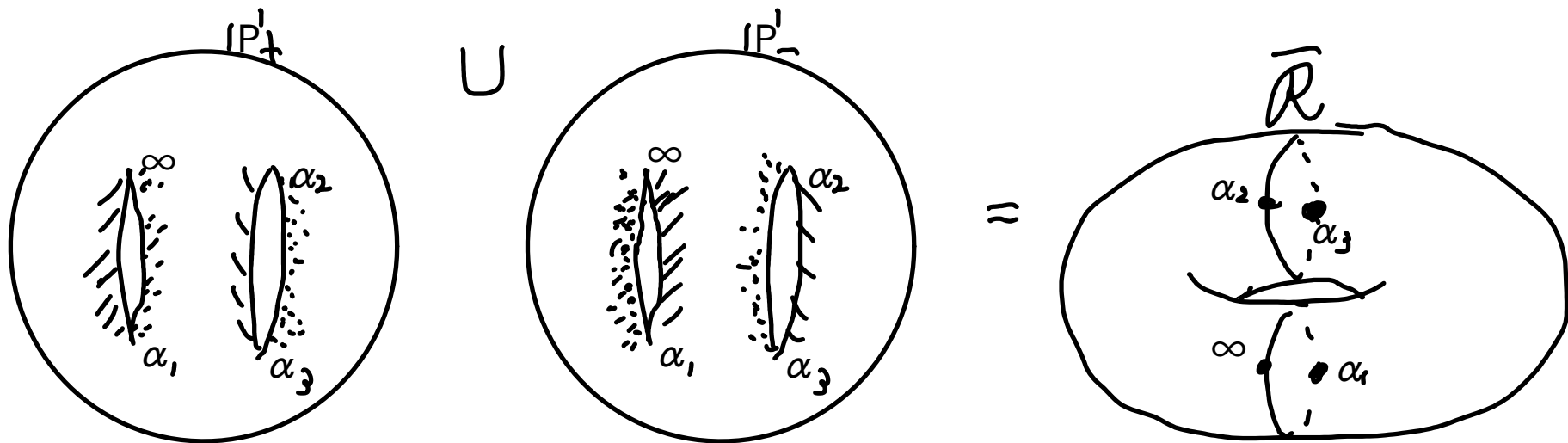
$$\overline{\{(z, w) \mid w^2 = (1-z^2)(1-k^2z^2)\}}^{\text{compactification}}, \quad k \in \mathbb{C}$$

as Riemann surfaces = 1-dim. complex manifold.

Exercise*: Prove this.

In particular, any elliptic curve is homeomorphic to a torus.

Gluing of \mathbb{P}^1 's when $\deg \varphi = 3$: $\varphi(z) = a(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$.



Cuts are $\infty\alpha_1$ and $\alpha_2\alpha_3$.