Elliptic Functions

Elliptic integrals over $\mathbb{R}$

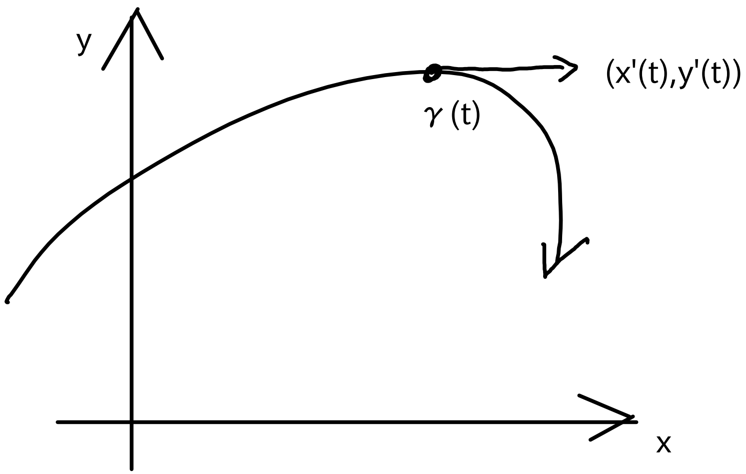
§1.1 Arc length of an ellipse

Everybody has learned that the length of a circle of radius a is $2\pi a$.

How can one *prove* this?

From the analysis course we know the formula for the arc length of a curve:

$\gamma : [a, b] \ni t \mapsto \gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$: a smooth curve.

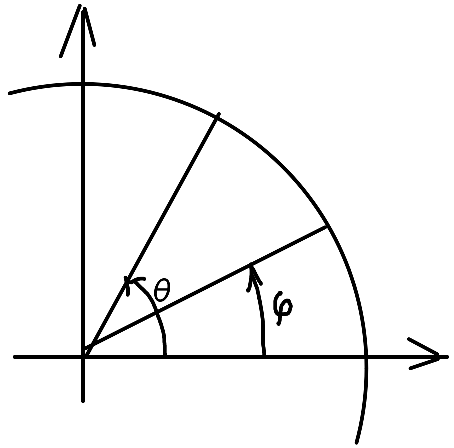


($x(t), y(t)$: C^1-class, i.e., $x'(t), y'(t)$ exist and are continuous.)

$$\implies \text{the length of } \gamma = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(The integrand $=$ speed of the moving point $(x(t), y(t))$.)

Parametrisation of an arc of a circle:



$$(x(\varphi), y(\varphi)) = (a\cos\varphi, a\sin\varphi), \qquad (\varphi \in [0, \theta]).$$

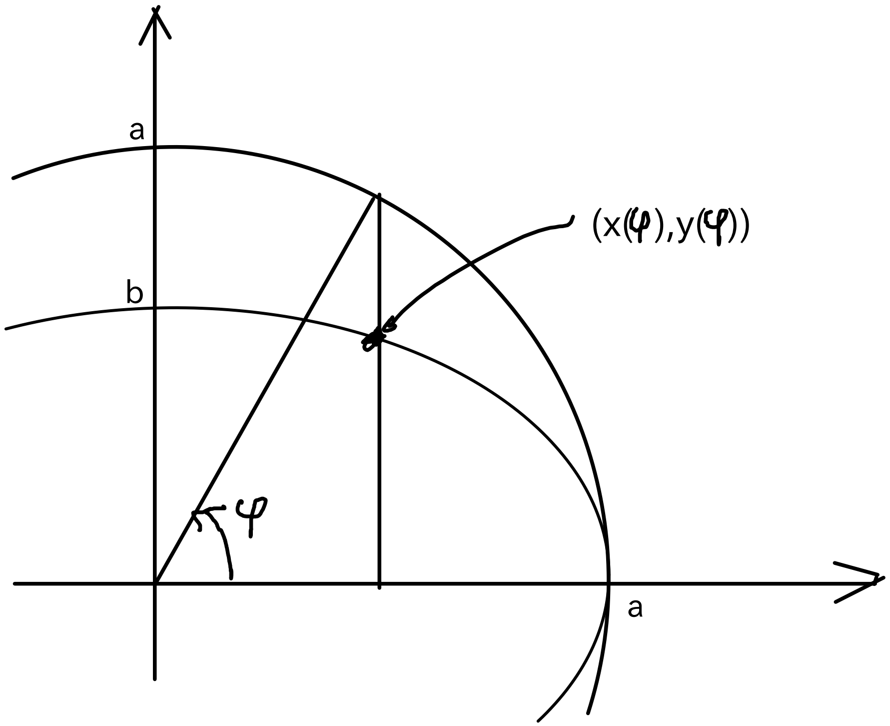
$$\begin{aligned}\text{The length of this arc} &= \int_0^\theta \sqrt{\left(\frac{d}{d\varphi} a\cos\varphi\right)^2 + \left(\frac{d}{d\varphi} a\sin\varphi\right)^2}\, d\varphi \\ &= \int_0^\theta \sqrt{a^2\sin^2\varphi + a^2\cos^2\varphi}\, d\varphi = a\theta.\end{aligned}$$

In particular the arc length of the circle $= a \times 2\pi$. $\square$

How about the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$?

Parametrisation of an arc of the ellipse:

$$(x(\varphi), y(\varphi)) = (a \sin\varphi, b\cos\varphi) \qquad (\varphi \in [0, \theta]).$$



$$
\begin{aligned}
\text{The length of this arc} &= \int_0^\theta \sqrt{\left(\frac{d}{d\varphi} a \sin\varphi\right)^2 + \left(\frac{d}{d\varphi} b \cos\varphi\right)^2}\, d\varphi \\
&= \int_0^\theta \sqrt{a^2 \cos^2\varphi + b^2 \sin^2\varphi}\, d\varphi \\
&= a \int_0^\theta \sqrt{1 - \frac{a^2 - b^2}{a^2} \sin^2\varphi}\, d\varphi \\
&= a \int_0^\theta \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi.
\end{aligned}
$$

$k := \sqrt{\dfrac{a^2 - b^2}{a^2}}$: *modulus* of the elliptic integral, *eccentricity* of the ellipse.

$$E(k,\theta) := \int_0^\theta \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

— incomplete *elliptic integral* of the second kind.

The length of the arc $(0 \leqq \varphi \leqq \theta) = aE\left(\sqrt{\frac{a^2-b^2}{a^2}}, \theta\right)$.

$$E(k) := E\left(k, \frac{\pi}{2}\right) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

— complete *elliptic integral* of the second kind.

The length of the ellipse $= 4aE\left(\sqrt{\frac{a^2-b^2}{a^2}}\right)$.

Except for the case $a = b$ (i.e., circles), such an integral cannot be expressed in terms of elementary functions.

(That's why we didn't learn this formula in schools!)

- Another expression of the elliptic integral of the second kind

Let us compute the arc length using the parametrisation:

$$(x, y(x)) = \left(x, b\sqrt{1 - \frac{x^2}{a^2}}\right). \qquad (x \in [0, a\cos\theta])$$

$$\begin{aligned}
\text{The arc length} &= aE(k, \theta) \\
&= \int_0^{a\sin\theta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{a\sin\theta} \sqrt{1 + \frac{b^2}{a^2}\frac{(x/a)^2}{1-(x/a)^2}}\, dx \\
&= a\int_0^{\sin\theta} \sqrt{\frac{1-k^2z^2}{1-z^2}}\, dz. \qquad (z = x/a)
\end{aligned}$$

In particular,

$$E(k,\theta) = \int_0^{\sin\theta} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz,$$

$$E(k) = \int_0^{1} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz.$$

Exercise:

(i) Find the arc length of other quadratic curves, i.e., of a parabola and a hyperbola. Which of them is expressed by an elliptic integral?

(ii) Express the arc length of the graph of $y = b\sin\dfrac{x}{a}$ in terms of the elliptic integral of the second kind. What arc correspond to $E(k)$?

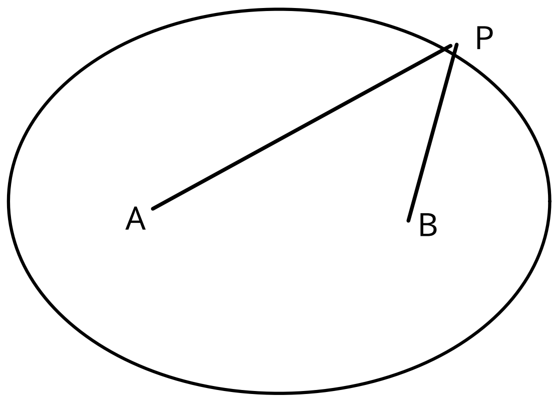
§1.2 Lemniscate and its arc length

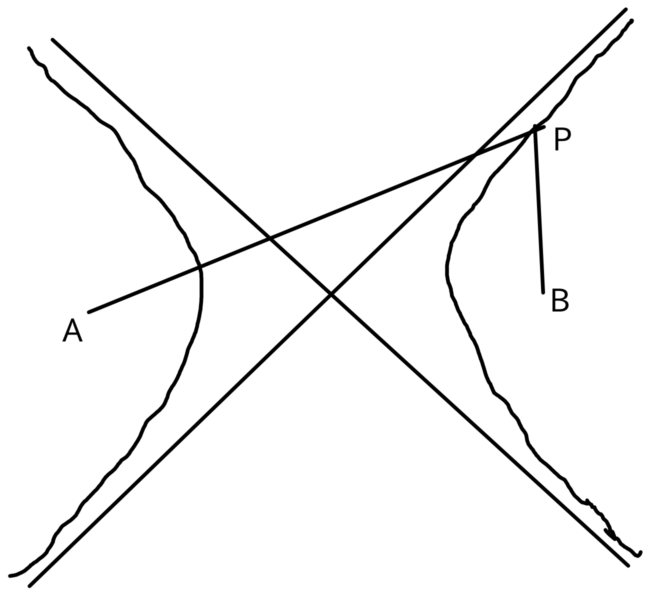
Fix two points A and B on a plane and a positive number l.

$$\text{Ellipse} = \{P \mid PA + PB = l\}.$$

$$\text{Hyperbola} = \{P \mid PA - PB = \pm l\}.$$

What is the curve defined by an equation $PA \times PB = \text{constant } (= l^2)$?





$P = (x, y) = (r\cos\varphi, r\sin\varphi)$, $A = (-a, 0)$, $B = (a, 0)$:

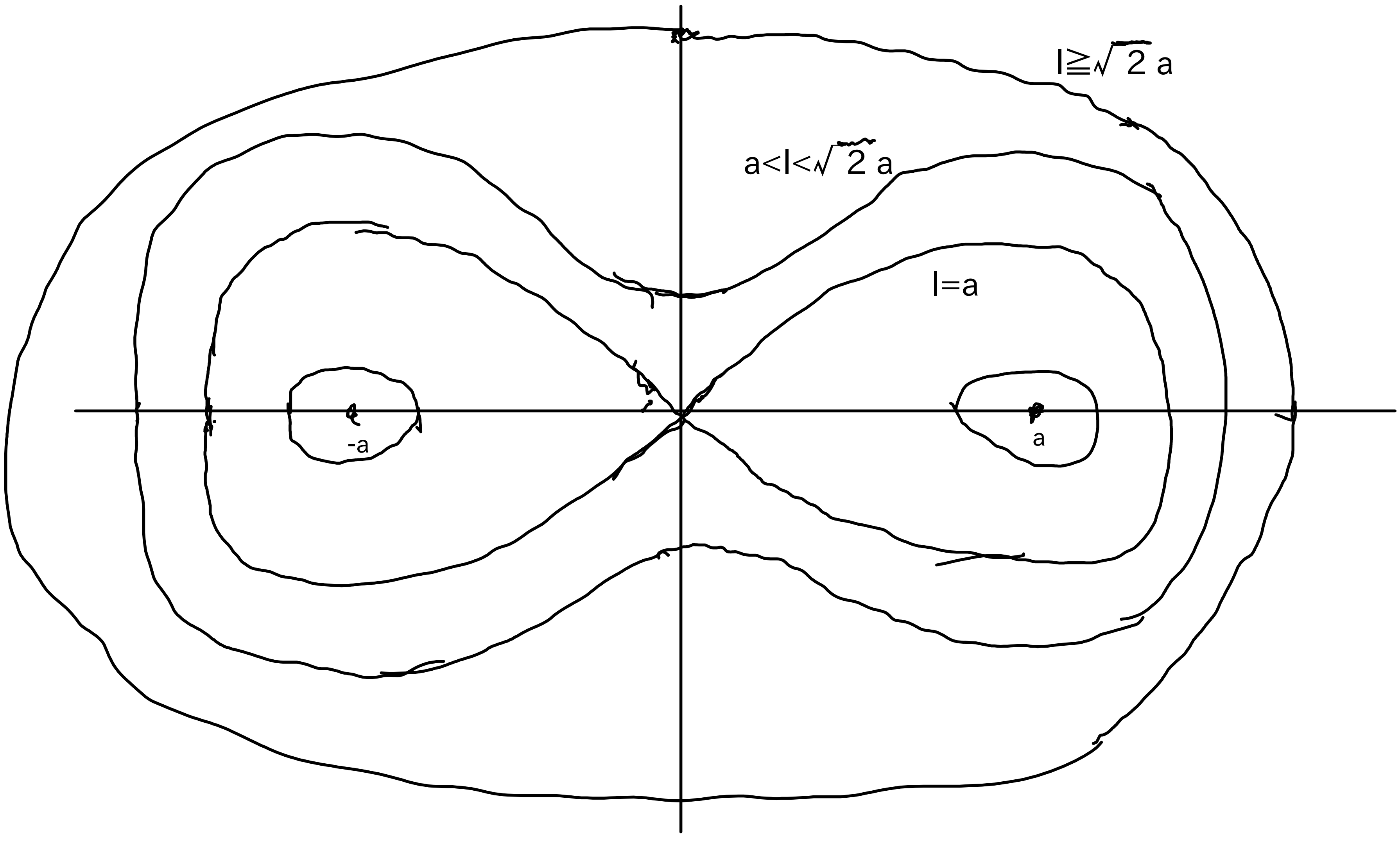
$$
\begin{aligned}
l^2 = PA \cdot PB &= \sqrt{(x+a)^2 + y^2}\sqrt{(x-a)^2 + y^2} \\
&= \sqrt{x^2 + y^2 + 2ax + a^2}\sqrt{x^2 + y^2 - 2ax + a^2} \\
&= \sqrt{r^2 + 2ar\cos\varphi + a^2}\sqrt{r^2 - 2ar\cos\varphi + a^2} \\
&= \sqrt{(r^2 + a^2)^2 - 4a^2r^2\cos^2\varphi} \\
&= \sqrt{r^4 + a^4 - 2a^2r^2\cos 2\varphi}.
\end{aligned}
$$

By squaring, we obtain an quartic equation:

$$\underline{\text{Cassini oval}}: \quad r^4 + a^4 - 2a^2r^2\cos 2\varphi = l^4.$$

The case $l = a$ is called the *lemniscate*.

Figures of Cassini oval and the lemniscate:



Equations for the lemniscate:

In polar coordinates:

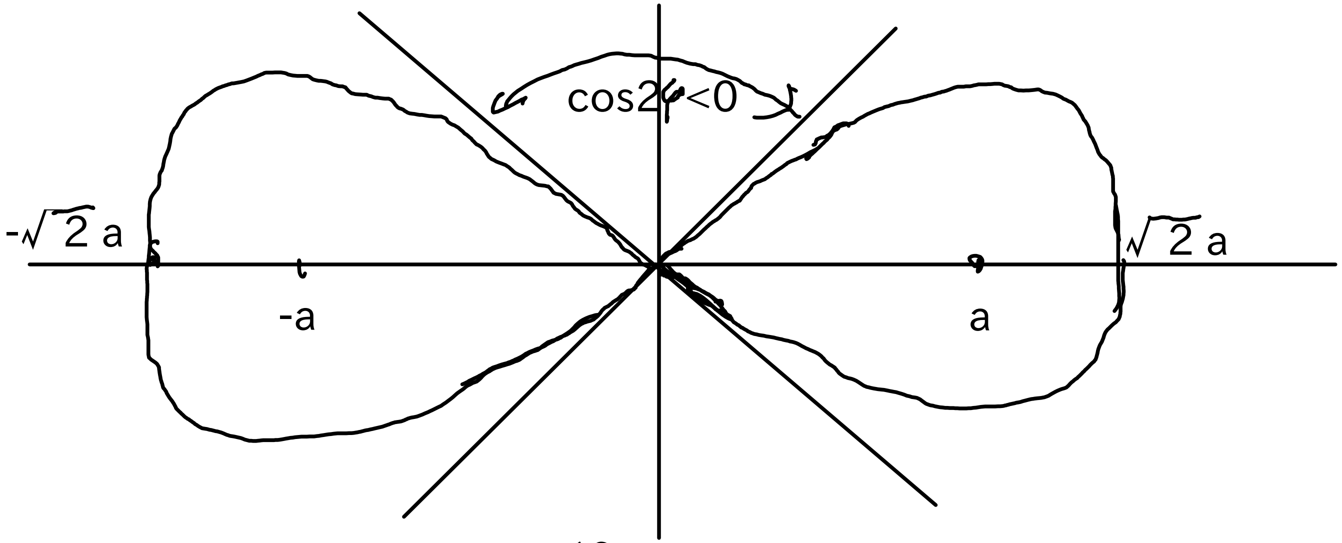
$$r^4 = 2a^2 r^2 \cos 2\varphi, \text{ i.e., } r^2 = 2a^2 \cos 2\varphi \text{ and } r = 0.$$

($\varphi \notin (\pi/4, 3\pi/4) \cup (5\pi/4, 7\pi/4)$, since $\cos 2\varphi$ should not be negative.)

In Cartesian coordinates:

$r^2 = x^2 + y^2$, $r^2 \cos 2\varphi = r^2 \cos^2 \varphi - r^2 \sin^2 \varphi = x^2 - y^2$, hence

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$



New parametrisation: $r = \sqrt{2}a\cos\psi$. (Note: $r^2 \leqq 2a^2$, i.e., $r \leqq \sqrt{2}a$.)

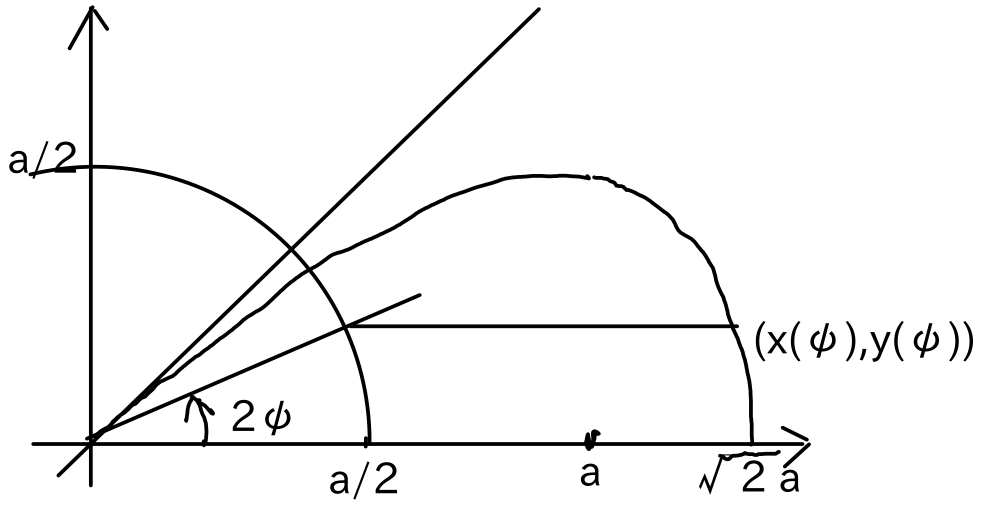
$$(x^2+y^2)^2 = r^4 = 4a^4\cos^4\psi, \text{ hence, } x^2 - y^2 = 2a^2\cos^4\psi.$$

Together with $x^2 + y^2 = 2a^2\cos^2\psi$,

$$x^2 = a^2\cos^2\psi(1+\cos^2\psi), \qquad y^2 = a^2\cos^2\psi(1-\cos^2\psi).$$

or, in the first quadrant ($x \geqq 0$, $y \geqq 0$; $0 \leqq \psi \leqq \pi/2$),

$$x = \sqrt{2}a\cos\psi\sqrt{1-\frac{1}{2}\sin^2\psi}, \qquad y = a\cos\psi\sin\psi = \frac{a}{2}\sin 2\psi.$$



Arc length of the lemniscate:

$$\frac{dx}{d\psi} = \sqrt{2}a\frac{\sin\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}}\left(-\frac{3}{2}+\sin^2\psi\right),$$

$$\frac{dy}{d\psi} = a(1-2\sin^2\psi).$$

$$\Longrightarrow \left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2 = \frac{a^2}{1-\frac{1}{2}\sin^2\psi}.$$

So, the arc length of the lemniscate is equal to

$$a\int_0^{\varphi} \frac{d\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}}.$$

$$F(k,\varphi) := \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

— incomplete *elliptic integral* of the first kind.

$$K(k) := F\left(k, \frac{\pi}{2}\right) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

— complete *elliptic integral* of the first kind.

The length of the arc $(0 \leqq \psi \leqq \varphi) = aF\left(\frac{1}{\sqrt{2}}, \varphi\right)$.

The length of the lemniscate $= 4aK\left(\frac{1}{\sqrt{2}}\right)$.

• <u>Another expression of the elliptic integral of the first kind</u>

Change the integration variable from ψ to $z := \sin\psi$:

$$dz = \cos\psi\, d\psi = \sqrt{1-z^2}\, d\psi$$

$$\Longrightarrow F(k,\varphi) = \int_0^{\sin\varphi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$