

Elliptic Functions

Complex Elliptic Integrals

§6.1 Complex elliptic integral of the first kind

Want: elliptic integrals $\int_C R(x, \sqrt{\varphi(x)}) dx$ with complex variables.

C : curve on the Riemann surface \mathcal{R} of $\sqrt{\varphi(z)}$,

or its compactification $\bar{\mathcal{R}}$ = the elliptic curve.

Let us begin with $\int \frac{dz}{\sqrt{\varphi(z)}}$, the elliptic integral of the first kind.

$$\omega_1 := \frac{dz}{\sqrt{\varphi(z)}} = \frac{dz}{w}.$$

We know that ω_1 is holomorphic on

$$\mathcal{R} = \bar{\mathcal{R}} \setminus \{\infty\} \text{ (deg } \varphi = 3) \text{ or } \mathcal{R} = \bar{\mathcal{R}} \setminus \{\infty_{\pm}\} \text{ (deg } \varphi = 4).$$

How about the neighbourhood of ∞_{\pm} ?

Assume $\text{deg } \varphi = 4$: $\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$.

(The case $\text{deg } \varphi = 3$ is similar.)

Recall:

- local parameter at $\infty_{\pm} = \xi = z^{-1}$.
- the equation of $\bar{\mathcal{R}}$ in the neighbourhood of ∞_{\pm} :

$$\eta^2 = a(1 - \alpha_0\xi)(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi),$$

where $\eta = wz^{-2}$.

- $\infty_{\pm} = (\xi = 0, \eta = \pm\sqrt{a} \neq 0)$.

Consequently,

- $d\xi = -z^{-2} dz$.
- $\omega_1 = \frac{dz}{w} = -\frac{d\xi}{\eta}$.
- $\eta(\xi) = \sqrt{a(1 - \alpha_0\xi)(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi)}$: holomorphic in ξ .
- $\eta(\xi) \neq 0$ at ∞_{\pm} .

$\implies \omega_1$ is holomorphic at ∞_{\pm} .

Conclusion: ω_1 is holomorphic everywhere on $\bar{\mathcal{R}}$.

Moreover, $\omega_1 \neq 0$ everywhere on $\bar{\mathcal{R}}$ ($\iff \omega_1 = \frac{dz}{w} = -\frac{d\xi}{\eta}$).

Exercise: Show that ω_1 is holomorphic everywhere and nowhere-vanishing in the case $\deg \varphi = 3$.

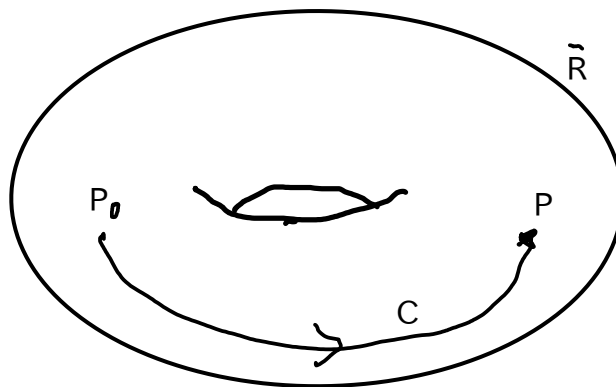
Fix $P_0 \in \bar{\mathcal{R}}$.

ω_1 is holomorphic on $\bar{\mathcal{R}}$.

$$\implies F(P) := \int_{P_0 \rightarrow P} \omega_1 = \int_{C: \text{contour from } P_0 \text{ to } P} \omega_1$$

is “locally” well-defined and holomorphic in P .

Figure of $\bar{\mathcal{R}}$ and C :

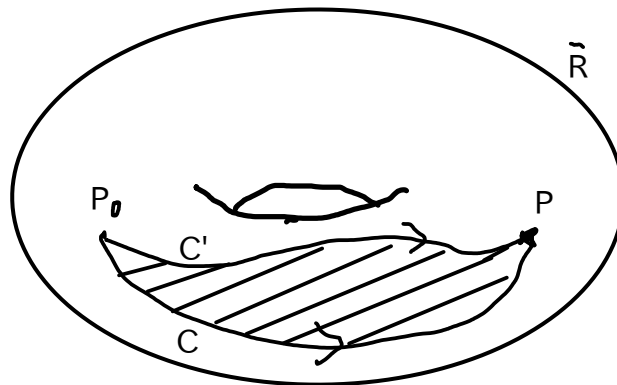


$\Leftrightarrow F(P)$ does not change by “small perturbation of C .”

Exactly speaking,

$$[C] - [C'] = 0 \text{ in } H_1(\bar{\mathcal{R}}, \mathbb{Z}) \implies \int_C \omega_1 = \int_{C'} \omega_1.$$

Figure: $[C] - [C'] = 0$.



Is $F(P)$ “globally” well-defined?

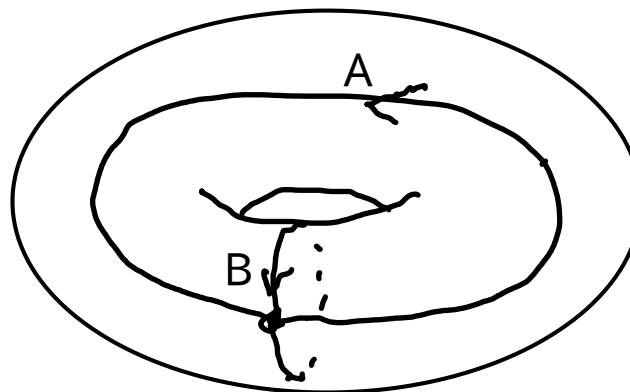
Need to know: How many “globally” different contours exist on $\bar{\mathcal{R}}$?

Answer from topology: $H_1(\bar{\mathcal{R}}, \mathbb{Z}) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$, which means:

for \forall closed curve C on $\bar{\mathcal{R}}$, $\exists! m, n \in \mathbb{Z}$, such that

$$[C] = m[A] + n[B] \text{ in } H_1(\bar{\mathcal{R}}, \mathbb{Z}).$$

Figure: A -cycle and B -cycle.



C_0, C_1 : curves from P_0 to P .

$\implies [C_1] - [C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

$$\int_{C_1} \omega_1 = \int_{C_0} \omega_1 + m \int_A \omega_1 + n \int_B \omega_1.$$

We call

$$\int_A \omega_1: \textit{A-period of 1-form } \omega_1, \quad \int_B \omega_1: \textit{B-period of 1-form } \omega_1.$$

Let us compute A - and B -periods for the case $\varphi(z) = (1 - z^2)(1 - k^2 z^2)$,
i.e.,

$$\omega_1 = \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

For simplicity, assume $k \in \mathbb{R}$, $0 < k < 1$.

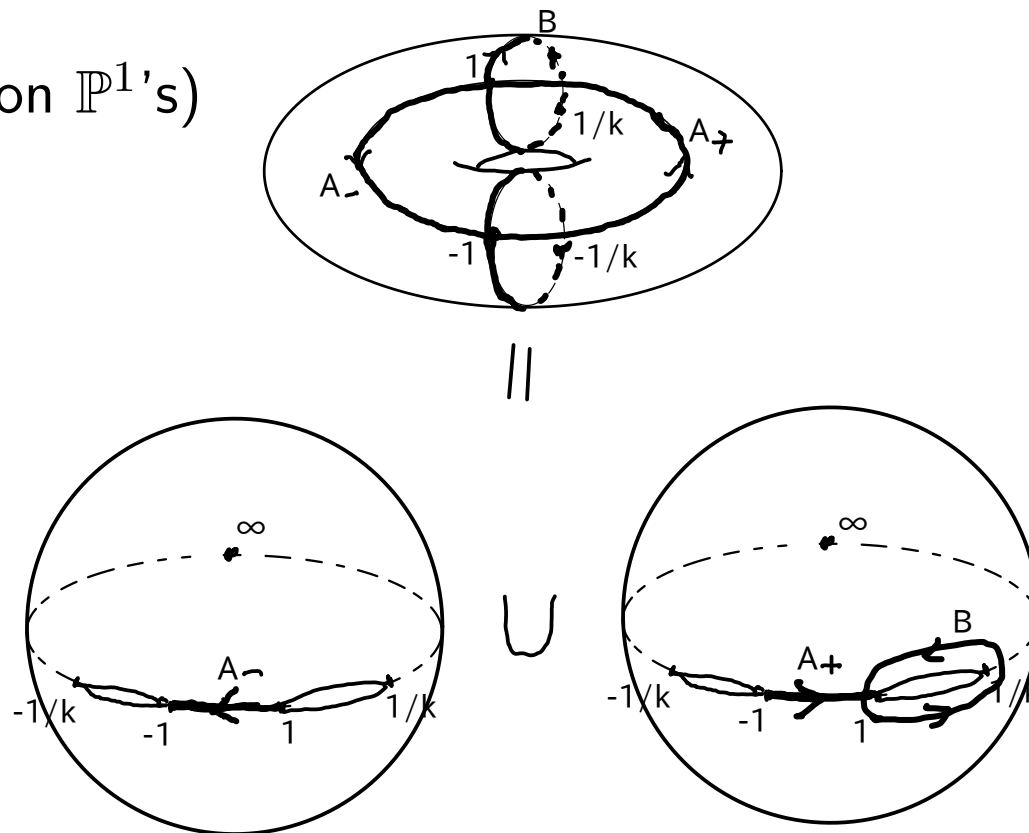
Recall the construction of $\bar{\mathcal{R}}$:

Two \mathbb{P}^1 's are glued together along cuts between two pairs of roots of $\varphi(z)$.

$$\text{roots of } \varphi(z) = \pm 1, \pm k^{-1}.$$

Cut \mathbb{P}^1 's along $[-k^{-1}, -1]$ and $[1, k^{-1}]$ and glue.

(Figure of A - and B -cycles on \mathbb{P}^1 's)



Periods of ω_1 :

$$\int_A \omega_1 = 4K(k), \quad \int_B \omega_1 = 2iK'(k),$$

where

- $K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$: complete elliptic integral of the first kind.
- $K'(k) := K(k')$, $k' := \sqrt{1-k^2}$ (supplementary modulus).

Proof:

$$\int_A \omega_1 = \int_{-1}^1 \frac{dx}{+\sqrt{(1-x^2)(1-k^2x^2)}} + \int_1^{-1} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}}$$

(Note: \pm because of branches.)

$$= 4 \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 4K(k).$$

$$\begin{aligned}
\int_B \omega_1 &= \int_1^{1/k} \frac{dx}{+\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{1/k}^1 \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}} \\
&= 2 \int_1^{1/k} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 2i \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}
\end{aligned}$$

Change of the variable: $x = \frac{1}{\sqrt{1-k'^2t^2}}$, i.e., $x^2 = \frac{1}{1-k'^2t^2}$,

$$dx = \frac{k'^2t}{(1-k'^2t^2)^{3/2}} dt, \quad (x^2-1)(1-k^2x^2) = \frac{k'^4t^2(1-t^2)}{(1-k'^2t^2)^2}.$$

Hence,

$$\int_B \omega_1 = 2i \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = 2i K(k') = 2i K'(k).$$

□

Remark:

- Signs of $\sqrt{}$ should be chosen carefully.
- For general $k \in \mathbb{C}$, the results are the same (analytic continuation).

Recall:

“A-period of $\frac{dz}{\sqrt{1-z^2}} = 2\pi = \text{period of } \sin(u).$ ”

Correspondingly,

A-period of $\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 4K(k) = \text{period of } \text{sn}(u)!$

What is the role of the B -period $2iK'(k)$ for $\text{sn}(u)$?

→ Another period of $\text{sn}(u)$, i.e., $\text{sn}(u)$ is doubly-periodic!

Details will be discussed later...

§6.2 Complex elliptic integral of the second kind

$$\int \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz = \int \frac{1 - k^2 z^2}{\sqrt{\varphi(z)}} dz, \quad \varphi(z) = (1 - z^2)(1 - k^2 z^2).$$

Corresponding Riemann surface = $\mathcal{R} = \{(z, w) \mid w^2 = \varphi(z)\}$ as before.

The compactification = $\bar{\mathcal{R}}$: elliptic curve.

$$\omega_2 := \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz = \frac{1 - k^2 z^2}{\sqrt{\varphi(z)}} dz = \frac{1 - k^2 z^2}{w} dz$$

is holomorphic on \mathcal{R} as ω_1 . (In particular, at $z = \pm 1, \pm k^{-1}$.)

How about at $\{\infty_{\pm}\} = \bar{\mathcal{R}} \setminus \mathcal{R}$?

Local coordinate at $\pm\infty$: $\xi = z^{-1}$.

$$\begin{aligned}\omega_2 &= \sqrt{\frac{1 - k^2\xi^{-2}}{1 - \xi^{-2}}} d(\xi^{-1}) = \sqrt{\frac{\xi^2 - k^2}{\xi^2 - 1}} \cdot (-\xi^{-2}) d\xi \\ &= -\xi^{-2}(\pm k + O(\xi^2)) d\xi = \left(\frac{\mp k}{\xi^2} + (\text{holomorphic at } \xi = 0) \right) d\xi.\end{aligned}$$

$\implies \omega_2$ has double poles at ∞_{\pm} without residues: $\text{Res}_{\infty_{\pm}} \omega_2 = 0$.

$$\implies G(P) := \int_{P_0 \rightarrow P} \omega_2 = \int_{C: \text{contour from } P_0 \text{ to } P} \omega_2 \quad \text{is}$$

- locally well-defined.
- holomorphic in P except at ∞_{\pm} .
- has a simple pole at ∞_{\pm} : $G(P) = \pm \frac{k}{\xi} + (\text{holomorphic at } \xi = 0)$.

Global multi-valuedness: same as the case of ω_1 .

C_0, C_1 : curves from P_0 to P .

$\implies [C_1] - [C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

$$\int_{C_1} \omega_2 = \int_{C_0} \omega_2 + m \int_A \omega_2 + n \int_B \omega_2.$$

$$\int_A \omega_2: \textit{A-period of } \omega_2, \quad \int_B \omega_2: \textit{B-period of } \omega_2.$$

Exercise: Express A - and B -periods of ω_2 in terms of the complete elliptic integral of the second kind.

§6.3 Complex elliptic integral of the third kind

$$\int \frac{dz}{(z^2 - a^2)\sqrt{\varphi(z)}}, \quad \varphi(z) = (1 - z^2)(1 - k^2 z^2).$$

$$\omega_3 := \frac{dz}{(z^2 - a^2)\sqrt{\varphi(z)}} = \frac{dz}{(z^2 - a^2)w}$$

is holomorphic on the elliptic curve (including ∞_{\pm}) except at four points:

$$(z, w) = (\pm a, \pm \sqrt{(1 - a^2)(1 - k^2 a^2)}).$$

These are *simple poles*.

Exercise: (i) Check these facts. (ii) Compute the residues at poles.

$$H(P) := \int_{P_0 \rightarrow P} \omega_3$$

multi-valued in the neighbourhood of simple poles because of the residue.

(And, of course, globally multi-valued.)

$\implies H(P)$ is a very complicated function.

Remark:

A meromorphic 1-form ω on a Riemann surface is called an *Abelian differential*. It is

- *of the first kind*, when ω is holomorphic everywhere.
- *of the second kind*, when the residue is zero at any pole.
- *of the third kind*, otherwise.

$\implies \omega_1$: the first kind, ω_2 : the second kind, ω_3 : the third kind.

(There are several different definitions; e.g.,

- “the third kind” has only simple poles’,
- “the second kind” has only one pole of order ≥ 2 ’, etc.)